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FROBENIUS PARTITION THEORETIC INTERPRETATION OF A FIFTH ORDER MOCK THETA FUNCTION

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ABSTRACT

Recently we gave two combinatorial interpretations of $F_1(q)$- a fifth order mock theta function by using (n+2) - color partitions and lattice paths. In this paper we give one more combinatorial meaning to the same mock theta function by using generalized Frobenius partitions. This results in a three - way combinatorial identity.
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INTRODUCTION
DEFINITIONS AND THE MAIN RESULTS

In his last letter dated 12 January, 1920 to Hardy, Ramanujan listed 17 functions which he called mock theta functions. He separated these 17 functions into three classes. First containing 4 functions of order 3, second containing 10 functions of order 5 and the third containing 3 functions of order 7.

Watson (1936) found three more functions of order 3 and two more of order 5 appear in the lost notebook (see Ramanujan, 1988). Mock theta functions of order 6 and 8 have also been studied by Andrews and Hickerson (1991) and Gordon and McIntosh (2000), respectively. For the definitions of mock theta functions and their order the reader is referred to Hardy *et al*. (1927).

A partition of a positive integer *n* is a non-increasing sequence of positive integers whose sum is *n*. 0 also has a partition called "empty partition". The rank of a partition is defined to be the largest part minus the number of its parts. Partition theoretic interpretations of some of the mock theta functions are found in the literature. For example, $\Psi(q)$, defined by (1.2) below, has been interpreted as generating function for partitions into odd parts without gaps (Fine, 1988). A survey of work done on mock theta functions is given in Andrews (1989). Recently Bringmann and Ono (2006) redefined mock theta functions as the holomorphic projection of weight 1/2 weak maass forms and used their ideas in solving the classical problem of obtaining formulas for $N_e(n)$ (resp. $N_0(n)$), the number of partitions of n with even (resp. odd) rank by showing the equivalence of this problem and the problem of deriving exact formulas for the coefficients $\alpha(n)$ of the series

$$f(q) = 1 + \sum_{n=1}^{\infty} \alpha(n) q^n = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(-q;q^2)_n}, \tag{1.1}$$

where $f(q)$ is the first mock theta function of order 3 in Ramanujan's list of 17 mock theta functions and

$$(a;q)_n = \prod_{i=o}^{\infty} \frac{(1-aq^i)}{(1-aq^{n+i})},$$

for any constant a.
The following four mock theta functions

$$\Psi(q) = \sum_{m=1}^{\infty} \frac{q^{m^2}}{(q;q^2)_m}, \tag{1.2}$$

$$F_0(q) = \sum_{m=0}^{\infty} \frac{q^{2m^2}}{(q;q^2)_m}, \tag{1.3}$$

$$\Phi_0(q) = \sum_{m=0}^{\infty} q^{m^2}(-q;q^2)_m, \tag{1.4}$$

and

$$\Phi_1(q) = \sum_{m=0}^{\infty} q^{(m+1)^2}(-q;q^2)_m, \tag{1.5}$$

where $\Psi(q)$ is of order 3 while the remaining three are of order 5, were interpreted combinatorially by Agarwal (2004), using n-color partitions. Later in 2005, he translated his results for lattice paths. For clarity we reproduce the results of Agarwal (2004, 2005). But first we recall the following definitions.

Definition 1.1 (Agarwal and Andrews, 1987) A partition with "$n+t$ copies of n," $t \geq 0$, is a partition in which a part of size n, $n \geq 0$, can come in $n+t$

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different colors denoted by subscripts: $n_1, n_2, \cdots, n_{n+t}$.

Thus, for example, the partitions of 2 with "$n+1$ copies of n" are

$$2_1,\quad 2_1+0_1,\quad 1_1+1_1,\quad 1_1+1_1+1_1,$$
$$2_2,\quad 2_2+0_1,\quad 1_2+1_1,\quad 1_1+1_1+0_1,$$
$$2_3,\quad 2_3+0_1,\quad 1_2+1_2,\quad 1_2+1_2+0_1.$$

Note that zeros are permitted if and only if t is greater than or equal to one. Also, in no partition are zeros permitted to repeat.

Definition 1.2 (Agarwal and Andrews, 1987) The weighted difference of two parts m_i, n_j, $m \geq n$ is defined by $m-n-i-j$ and denoted by $((m_i - n_j))$.
Next we describe the lattice paths which we shall be considering in this paper (Agarwal and Bressoud, 1989). All paths will be of finite length lying in the first quadrant . They will begin on the y-axis and terminate on the x-axis. Only three moves are allowed at each step:
northeast: from (i, j) to $(i+1, j+1)$
southeast: from (i, j) to $(i+1, j-1)$, only allowed if $j > 0$
horizontal: from $(i, 0)$ to $(i+1, 0)$, only allowed along x-axis

All our lattice paths are either empty or terminate with a southeast step: from $(i, 1)$ to $(i+1, 0)$.

The following terminology will be used in describing lattice paths:

Peak: Either a vertex on the y-axis which is followed by a southeast step or a vertex preceded by a northeast step and followed by a southeast step.

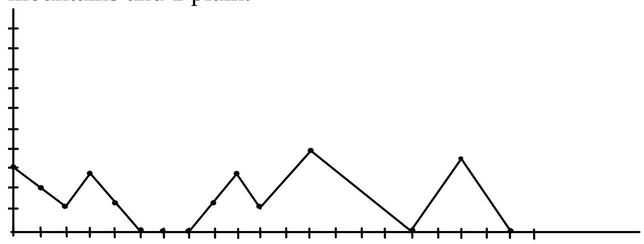
Valley: A vertex preceded by a southeast step and followed by a northeast step. Note that a southeast step followed by a horizontal step followed by a northeast step does not constitute a valley.

Mountain: A section of the path which starts on either the x-axis or y-axis, which ends on the x-axis, and which does not touch the x-axis anywhere in between the end points. Every mountain has at least one peak and may have more than one.

Plain: A section of the path consisting of only horizontal steps which starts either on the y-axis or at a vertex preceded by a southeast step and ends at a vertex followed by a northeast step.

The **Height** of a vertex is its y-coordinate. The **Weight** of a vertex is its x-coordinate. The **Weight** of a path is the sum of the weights of its peaks.

Example: The following path has 5 peaks, 3 valleys, 3 mountains and 1 plain.



Definition 1.3 (Andrews, 1984) A two rowed array of non-negative integers
$$\begin{pmatrix} a_1 & a_2 & \cdots & a_r \\ b_1 & b_2 & \cdots & b_r \end{pmatrix},$$
$a_1 \geq a_2 \geq \cdots \geq a_r \geq 0$, $b_1 \geq b_2 \geq \cdots \geq b_r \geq 0$,
is known as a generalized Frobenius partition or more simply an F-partition of n if $n = r + \sum_{i=1}^{r} a_i + \sum_{i=1}^{r} b_i$

For example, n=28=4+(6+5+2+0)+(5+3+2+1) and the corresponding Frobenius notation is $\begin{pmatrix} 6 & 5 & 2 & 0 \\ 5 & 3 & 2 & 1 \end{pmatrix}$.

It was proved by Agarwal (2004) that the mock theta functions (1.2) - (1.5) have their combinatorial interpretations in the following theorems, respectively:

Theorem 1.1. For $\nu \geq 1$, let $A_1(\nu)$ denote the number of n-color partitions of ν such that even parts appear with even subscripts and odd with odd. For some k, k_k is a part, and the weighted difference of any two consecutive parts is 0. Then
$$\sum_{\nu=1}^{\infty} A_1(\nu) q^{\nu} = \Psi(q). \tag{1.6}$$

Theorem 1.2. For $\nu \geq 0$, let $A_2(\nu)$ denote the number of n-color partitions of ν such that even parts appear with even subscripts and odd with odd greater than 1, for some k, k_k is a part, and the weighted difference of any two consecutive parts is 0. Then
$$\sum_{\nu=0}^{\infty} A_2(\nu) q^{\nu} = F_0(q). \tag{1.7}$$

Theorem 1.3. For $\nu \geq 0$, let $A_3(\nu)$ denote the number of n-color partitions of ν such that only the first copy of the odd parts and the second copy of the even parts are used, that is, the parts are of the type $(2k-1)_1$ or $(2k)_2$, the minimum part is 1_1 or 2_2 and the weighted difference of any two consecutive part is 0. Then
$$\sum_{\nu=0}^{\infty} A_3(\nu) q^{\nu} = \Phi_0(q). \tag{1.8}$$

Theorem 1.4. For $\nu \geq 1$, let $A_4(\nu)$ denote the number of n-color partitions of ν such that only the first copy of the odd parts and the second copy of the even parts are used, the minimum part is 1_1 and the weighted difference of any two consecutive part is 0. Then

$$\sum_{\nu=0}^{\infty} A_4(\nu)q^\nu = \Phi_1(q). \tag{1.9}$$

Later Agarwal (2005) translated Theorems (1.1) - (1.4) for lattice paths as follows:

Theorem 1.5. For $\nu \geq 1$, let $B_1(\nu)$ denote the number of lattice paths of weight ν which start from (0,0), have no valley above height 0 and no plain. Then

$$\sum_{\nu=1}^{\infty} B_1(\nu)q^\nu = \Psi(q). \tag{1.10}$$

Theorem 1.6. For $\nu \geq 0$, let $B_2(\nu)$ denote the number of lattice paths of weight ν which start from (0,0), have no valley above height 0, no plain and the height of each peak is≥ 2. Then

$$\sum_{\nu=0}^{\infty} B_2(\nu)q^\nu = F_0(q). \tag{1.11}$$

Theorem 1.7. For $\nu \geq 0$, let $B_3(\nu)$ denote the number of lattice paths of weight ν which start from (0,0), have no valley above height 0, no plain, the height of each peak of odd weight is 1 while that of even weight is 2. Then

$$\sum_{\nu=0}^{\infty} B_3(\nu)q^\nu = \Phi_0(q). \tag{1.12}$$

Theorem 1.8. For $\nu \geq 1$, let $B_4(\nu)$ denote the number of lattice paths of weight ν which start from (0,0), have no valley above height 0, no plain, the height of each peak of odd weight is 1 while that of even weight is 2 and the weight of the first peak is 1. Then

$$\sum_{\nu=0}^{\infty} B_4(\nu)q^\nu = \Phi_1(q). \tag{1.13}$$

Very recently, we gave two combinatorial interpretations of $F_1(q)$ - a fifth order mock theta function, defined by

$$F_1(q) = \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q;q^2)_{n+1}}, \tag{1.14}$$

in the following form:

Theorem 1.9. For $\nu \geq 0$, let $B(\nu)$ denote the number of partitions of ν with "$n+2$ copies of n" in which even parts appear with even subscripts and odd with odd greater than 1. For some i, i_{i+2} is a part and the weighted difference of any two consecutive parts is zero. Let $C(\nu)$ denote the number of lattice paths of weight ν which start at (0,2), have no valley above height 0, no plain, and for which the height of each peak is ≥ 2. Then

$$\sum_{\nu=0}^{\infty} B(\nu)q^\nu = \sum_{\nu=0}^{\infty} C(\nu)q^\nu = F_1(q). \tag{1.15}$$

Theorem 1.9 yields the following combinatorial identity

$$B(\nu) = C(\nu), \quad for \;\; all \;\; \nu. \tag{1.16}$$

Note. Theorem 1.9 will appear in the Centenary Volume of the Journal of the Indian Mathematical Society, 2007. Here we propose to prove the following theorem which extends Theorem 1.9:

Theorem 1.10. For $\nu \geq 0$, let $A(\nu)$ denote the number of F-partitions of ν such that

(1.a) $a_r = \; 0 \; or \; 2$,
(1.b) $a_i \leq b_i + 1$, and
(1.c) $a_i = b_{i+1} + 3$.

Let $B(\nu)$ denote the number of $(n+2)$ - color partitions of ν such that
(1.d) even parts appear with even subscripts and odd with odd greater than 1,
(1.e) the weighted difference of any two consecutive parts is 0, and
(1.f) for some i, i_{i+2} is a part.
Then

$$A(\nu) = B(\nu), \quad for \;\; all \;\; \nu.$$

Note. $B(\nu)$ of Theorem 1.10 is the same as in Theorem 1.9.

We give the detail proof of Theorem 1.10 in our next section and conclude in the last section by posing an open problem.

Proof of the Theorem 1.10

We establish a 1-1 correspondence between the F-partitions enumerated by $A(\nu)$ and the $(n+2)$ - color partitions enumerated by $B(\nu)$. We do this by mapping each column $\begin{pmatrix} a \\ b \end{pmatrix}$ of the F-partition to a single part m_i of an $(n+2)$- color partition enumerated by $B(\nu)$. The mapping ϕ is

$$\phi: \begin{pmatrix} a \\ b \end{pmatrix} \to (a+b+1)_{b-a+3}, \tag{2.1}$$

and the inverse mapping ϕ^{-1} is given by

$$\phi^{-1}: m_i = \begin{pmatrix} (m-i+2)/2 \\ (m+i-4)/2 \end{pmatrix}. \tag{2.2}$$

Now suppose we have any two adjacent columns $\begin{pmatrix} a \\ b \end{pmatrix}$ and $\begin{pmatrix} c \\ d \end{pmatrix}$ in an F- partition enumerated by $A(\nu)$ with

$$\phi: \begin{pmatrix} a \\ b \end{pmatrix} = m_i \text{ and } \phi: \begin{pmatrix} c \\ d \end{pmatrix} = n_j.$$

Then since

$$\begin{pmatrix} a \\ b \end{pmatrix} \to (a+b+1)_{b-a+3} = m_i \text{ and } \begin{pmatrix} c \\ d \end{pmatrix} \to (c+d+1)_{d-c+3} = n_j,$$

We have $((m_i - n_j)) = m - n - i - j$

$$= (a+b+1) - (c+d+1) - (b-a+3) - (d-c+3)$$
$$= 2(a-d) - 6. \qquad (2.3)$$

Clearly (2.3) and (1.c) imply (1.e).
Also (2.1), (1.a) and (1.b) imply (1.d) as (a+b+1) and (b-a+3) are of same parity.
Now if $a_r = 0$ then $\phi \begin{pmatrix} a_r \\ b_r \end{pmatrix} = (b_r+1)_{b_r+3}$ which is of the form i_{i+2}, and if $a_r = 2$, then $\phi \begin{pmatrix} a_r \\ b_r \end{pmatrix} = (b_r+3)_{b_r+1}$. In this case we consider a "phantom" column $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$, as the last column. Since $\phi \begin{pmatrix} 0 \\ -1 \end{pmatrix} = 0_2$, we see that (1.f) holds and the parts $(b_r+3)_{b_r+1}$ and 0_2 satisfy (1.e). It is worthwhile to mention here that the "phantom" column is dropped from the full Frobenius symbol.

To see the reverse implication, we consider the inverse images of two consecutive parts $m_i,\ n_j$ of an $(n+2)$-color partition enumerated by $B(\nu)$

$$\phi^{-1} : m_i = \begin{pmatrix} (m-i+2)/2 \\ (m+i-4)/2 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

and

$$\phi^{-1} : n_j = \begin{pmatrix} (n-j+2)/2 \\ (n+j-4)/2 \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix}$$

that is,

$$a = (m-i+2)/2 \qquad (2.4)$$

$$b = (m+i-4)/2 \qquad (2.5)$$

$$c = (n-j+2)/2 \qquad (2.6)$$

$$d = (n+j-4)/2 \qquad (2.7)$$

and so

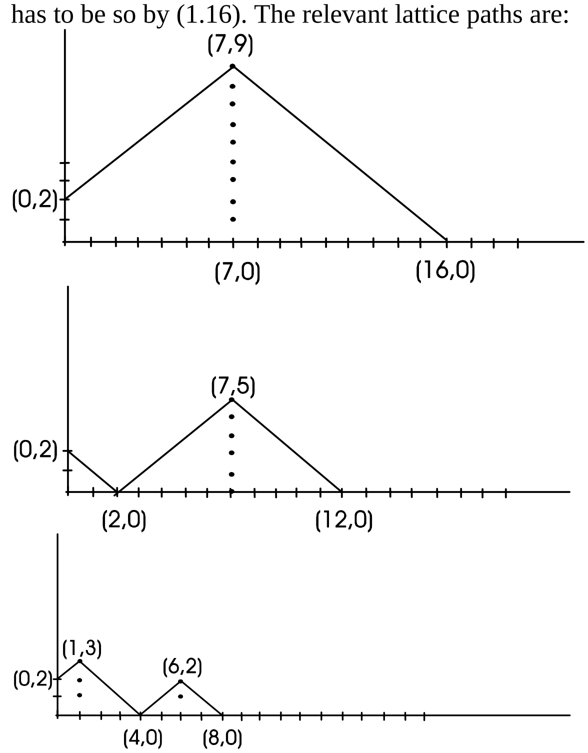
$$b - a = i - 3 \qquad (2.8)$$
$$d - c = j - 3 \qquad (2.9)$$
$$a - d = \frac{1}{2}((m_i - n_j)) + 3 \qquad (2.10)$$

(2.10) and (1.e) imply (1.c).
(2.8) and (2.9) imply (1.b).
(1.f) implies that there is a column of the form $\begin{pmatrix} 0 \\ i-1 \end{pmatrix}$.
Such a column has to be the last in the F - partition and i_{i+2} must be the smallest part of its partition, since if $i_{i+2} > n_j$ then
$((i_{i+2} - n_j)) = -2 - n - j < 0$.
Also 0_2 is allowed to be a part in an $(n+2)$- color partition enumerated by $B(\nu)$. 0_2 corresponds to a "phantom" column $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$, which is dropped from the corresponding F- partition. This in view of (1.e) implies $a_r = 2$. Otherwise, if i_{i+2} $(i \neq 0)$ is the last part in $(n+2)$- color partition then using (2.2) we see that it corresponds to a column $\begin{pmatrix} 0 \\ i-1 \end{pmatrix}$ which implies $a_r = 0$.
This completes the proof of the Theorem 1.10.

To illustrate the bijection we have constructed we close this section with the example for ν=7 shown in the following Table 1

F-partitions enumerated by $A(7)$	Image under ϕ
$\begin{pmatrix} 0 \\ 6 \end{pmatrix}$	7_9
$\begin{pmatrix} 2 \\ 4 \end{pmatrix}$	$7_5 + 0_2$
$\begin{pmatrix} 3 & 0 \\ 2 & 0 \end{pmatrix}$	$6_2 + 1_3$

Furthermore, we see that C (7) is also equal to 3, which has to be so by (1.16). The relevant lattice paths are:



CONCLUSION

Theorems (1.9) and (1.10) lead to the following 3 - way combinatorial identity

$$A(\nu) = B(\nu) = C(\nu). \qquad (3.1)$$

Obviously, (3.1) induces three combinatorial identities. While the identity $B(\nu) = C(\nu)$ is (1.16) given above, the other two viz., $A(\nu) = B(\nu)$ and $A(\nu) = C(\nu)$ are new. Agarwal (2004, 2005) gave combinatorial interpretations of four mock theta functions defined by (1.2)-(1.5) above by using colored partitions and lattice paths. Recently, Agarwal and Narang have given combinatorial interpretations of the same mock theta functions by using F - partitions. Their results have been accepted for publications in ARS Combinatoria. Very recently Agarwal and Rana succeeded in interpreting one more mock theta function of order five defined by (1.14) above combinatorially by using $(n+2)$-color partitions and lattice paths. Their results will appear in the Centenary Volume of the Journal of the Indian Mathematical Society, 2007. In this paper we have interpreted this mock theta function combinatorially, in terms of F - partitions. It would be of interest to interpret the other mock theta functions also by using the methods of these papers.

REFERENCES

Agarwal, AK. 2005. Lattice paths and mock theta functions. Proceedings of the 6th Int. Conf., SSFA. 6:95-102.

Agarwal, AK. 2004. *n*-color partition theoretic interpretations of some mock theta functions. Electron. J. Combin. 11(1) Note 14, 6pp. (electronic).

Agarwal, AK. and Andrews, GE. 1987. Rogers-Ramanujan identities for partitions with "N copies of N". J. Combin. Theory Ser. A. 45(1):40-49.

Agarwal, AK. and Bressoud, DM. 1989. Lattice paths and multiple basic hypergeometric series. Pacific J. Math. 136(2):209-228.

Andrews, GE. 1989. Mock theta functions in Theta Functions Bowdoin, Part 2. Proceedings of Symposia in Pure Mathematics, American Mathematical Society. Providence, Rhode Island. (49):283-296.

Andrews, GE. 1949. Generalized Frobenius partitions. Mem. Amer. Math. Soc. 49(301):iv+pp 44.

Andrews, GE. and Hickerson, D. 1991. Ramanujan's "lost" Notebook VII: The sixth order mock theta functions. Adv. Math. 89:60-105.

Bringmann, K. and Ono, K. 2006. The f(q) mock theta function conjecture and partition ranks. Invent. math. 165: 243-266.

Fine, NJ. 1988. Basic Hypergeometric Series and Applications. Mathematical Surveys and Monographs. AMS. 27.

Gordon, B. and McIntosh, RJ. 2000. Some eight order mock theta functions. J. London Math. Soc. (2):321-335.

Hardy, GH., SeshuAiyar, PV. and Wilson, BM. 1927. Collected papers of Srinivasa Ramanujan. Cambridge Univ. Press.

Ramanujan, S. 1988. The Lost Notebook and other Unpublished Papers. Narosa Publishing House. New Delhi.

Watson, GN. 1936. The Final problem: an account of the mock theta functions. J. London math. Soc. 11:55-80.