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# $\gamma_0$-COMPACT , $\gamma^s$-REGULAR AND $\gamma^s$-NORMAL SPACES

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## ABSTRACT

We define and study the properties of $\gamma^s$ -regular and $\gamma^s$ -normal spaces. We also continue studying $\gamma_0$-compact spaces defined in Ahmad and Hussain (2006).

**Keywords**: γ-closed(open), γ-closure, γ-regular (open), (γ,β)-continuous (closed, open) functions, $\gamma_0$-compact, $\gamma^s$-regular spaces and $\gamma^s$-normal spaces.
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## INTRODUCTION

Kasahara (1979) defined an operation α on topological spaces. He introduced and studied α-closed graphs of a function. Jankovic (1983) defined α- closed setss and further worked on functions with α- closed graphs. Ogata (1991) introduced the notions of $\gamma$-$T_i$ ,i = 0,1/2,1,2; and studied some topological properties. Rehman and Ahmad (1992) [respt. Ahmad and Rehman 1993] defined and investigated several properties of γ-interior, γ-exterior, γ-closure and γ-boundary points in topological spaces (respt. in product spaces), and studied the characterizations of (γ,β)-continuous mappings initiated by Ogata (1991). Ahmad and Hussain (2003) continued studying the properties of γ-operations on topological spaces introduced by Kasahara (1979). Ahmad and Hussain (2005) defined γ-nbd, γ-nbd base at x, γ-closed nbd, γ-limit point, γ-isolated point, γ-convergent point and $\gamma^*$-regular spaces and discussed their several properties. They further established the properties of (γ,β)-continuous, (γ,β)- open functions and $\gamma$-$T_2$ spaces.

In this paper, we continue studying $\gamma_0$-compact spaces defined in Ahmad and Hussain (2006) and study its properties. We also define and study some properties of $\gamma^s$-regular and $\gamma^s$-normal spaces.

First, we recall some definitions and results used in this paper. Hereafter we shall write spaces in place of topological spaces.

**Definition** (Ogata, 1991). Let (X,τ) be a space. An operation $\gamma : \tau \to P(X)$ is a function from τ to the power set of X such that $V \subseteq V^\gamma$ , for each $V \in\tau$, where $V^\gamma$ denotes the value of γ at V. The operations defined by γ(G) = G, γ(G) = cl(G) and γ(G) = intcl(G) are examples of operation γ.

**Definition** (Ogata, 1991). Let $A \subseteq X$. A point $a \in A$ is said to be γ-interior point of A iff there exists an open nbd N of a such that $N^\gamma \subseteq A$ and we denote the set of all such points by $int_\gamma$ (A).
Thus

$$int_\gamma (A) = \{ x \in A : x \in N \in\tau \text{ and } N^\gamma \subseteq A \} \subseteq A.$$

Note that A is γ-open ( Ogata, 1991) iff $A = int_\gamma (A)$. A set A is called γ- closed (Ogata, 1991) iff X−A is γ-open.

**Definition** (Rehman and Ahmad, 1992). A point $x \in X$ is called a γ-closure point of $A \subseteq X$, if $U^\gamma \cap A \neq \emptyset$, for each open nbd U of x. The set of all γ-closure points of A is called γ-closure of A and is denoted by $cl_\gamma(A)$. A subset A of X is called γ-closed, if $cl_\gamma(A) \subseteq A$.

Note that $cl_\gamma(A)$ is contained in every γ-closed superset of A.

**Definition 1**. An operation $\gamma : \tau \to P(X)$ is said to be strictly regular, if for any open nbds U , V of $x \in X$, there exists an open nbd W of x such that $U^\gamma \cap V^\gamma = W^\gamma$.

**Definition 2**. An operation $\gamma : \tau \to P(X)$ is said to be γ-open, if $V^\gamma$ is γ-open for each $V \in \tau$.

**Example 1**. Let X={a,b,c}, τ ={ ∅, X, {a}, {b},{a,b}}.

Define an operation $\gamma : \tau\to P(X)$ by γ(A) = intcl(A).

Clearly the γ-open sets are only ∅, X, {a}, {b},{a,b}. It is easy to see that γ is strictly regular and γ-open on X.

**Example 2**. Let X={a,b,c}, τ ={ ∅, X, {a}, {b},{a,b}}.
Define an operation $\gamma : \tau\to P(X)$ by γ(A) = cl(A).

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Clearly the $\gamma$-open sets are only $\emptyset$, X. It is easy to see that $\gamma$ is strictly regular but not $\gamma$-open on X.

## 1. $\gamma_0$-compact spaces

**Definition** (Ahmad and Hussain, 2006). A space X is said to be $\gamma_0$-compact, if for every cover $\{V_i : i \in I\}$ of X by $\gamma$-open sets of X, there exists a finite subset $I_0$ of I such that

$$X = \bigcup_{i \in I_0} cl_\gamma(V_i).$$

Then the following characterization of a $\gamma_0$-compact space is immediate:

**Theorem** (Ahmad and Hussain, 2006). A space X is $\gamma_0$-compact iff every class of $\gamma$-open and $\gamma$- closed sets with empty intersection has a finite subclass with empty intersection.

**Definition** (Ogata, 1991). A space X is said to be $\gamma$-$T_2$ space. If for each pair of distinct points x, y in X, there exist open sets U , V such that $x \in U$, $y \in V$ and $U^\gamma \cap V^\gamma = \emptyset$.

**Definition (**Ogata, 1991). An operation $\gamma$ is said to be regular, if for any open nbds U ,V of $x \in X$, there exists an open nbd W of x such that $U^\gamma \cap V^\gamma \supseteq W^\gamma$.

**Thorem 1**. Let X be a $\gamma$-$T_2$ space and suppose that C be a $\gamma_0$-compact subset of X and $x \in X - C$, then there are open sets $U_x$ and $V_x$ in X such that $x \in U^\gamma_x$ and $C \subseteq V^\gamma_x$ and $U^\gamma_x \cap V^\gamma_x = \emptyset$, where $\gamma$ is regular and $\gamma$-open.

**Proof**. Suppose that C is $\gamma_0$-compact subset of X and $x \in X - C$. For each $y \in C$ , $y \neq x$. Since X is $\gamma$-$T_2$ , there are open sets $U_{xy}$ and $V_y$ containing x and y respectively such that $U^\gamma_{xy} \cap V^\gamma_y = \emptyset$. Now, let $\{ V^\gamma_y \cap C : y \in C \}$ be $\gamma$-open cover of C. Since C is $\gamma_0$-compact , then $\gamma$-open cover has a finite subset $\{V^\gamma y_1 \cap C, V^\gamma y_2 \cap C, \dots, V^\gamma y_n \cap C\}$ such that

$$C = \bigcup_{i=1}^{n} cl_\gamma( V^\gamma y_i \cap C ).$$

Let $U^\gamma y_1, U^\gamma y_2, \dots, U^\gamma y_n$ be the corresponding $\gamma$-open sets containing x. Take

$$U^\gamma_x = (\bigcap_{i=1}^{n} cl_\gamma(U^\gamma xy_i))$$

and $$V^\gamma_x = (\bigcup_{i=1}^{n} cl_\gamma(V^\gamma xy_i)),$$

then $x \in U^\gamma_x$ and $C \subseteq V^\gamma_x$ . Where $U^\gamma_x$ and $V^\gamma_x$ are $\gamma$-closed , since $\gamma$ is regular.

Also $$U^\gamma_x \cap V^\gamma_x = (\bigcap_{i=1}^{n} cl_\gamma(U^\gamma xy_i)) \cap (\bigcup_{i=1}^{n} cl_\gamma(V^\gamma y_i))$$

$$= \bigcap_{i=1}^{n}(\bigcup_{i=1}^{n} cl_\gamma(U^\gamma xy_i) \cap cl_\gamma(V^\gamma y_i))$$

$$= \bigcap_{i=1}^{n}(\bigcup_{i=1}^{n} cl_\gamma(U^\gamma xy_i \cap V^\gamma y_i))$$

[ $\gamma$ is regular ( Rehman and Ahmad, 1992) ]

$$= \bigcap_{i=1}^{n}(\bigcup_{i=1}^{n} cl_\gamma(\emptyset)) = \emptyset.$$

**Theorem 2**. Let X be a $\gamma$-$T_2$ space. Then every $\gamma_0$-compact subset A of X is $\gamma$-closed, where $\gamma$ is regular and $\gamma$-open.

**Proof**. Let X be a $\gamma$-$T_2$ space and A be a $\gamma_0$-compact subset of X. We show that X − A is $\gamma$-open. For this, let $x \in X - A$. Then $y \in A$ gives $x \neq y$. Since X is $\gamma$-$T_2$ , there are open sets $U_{xy}$ and $U_y$ in X containing x and y respectively such that $U^\gamma_{xy} \cap U^\gamma_y = \emptyset$.

Let the collection $\{ U^\gamma_y \cap A : y \in A \}$ is a cover of A by $\gamma$-open and $\gamma$-closed sets of A.

Since A is $\gamma_0$-compact, there is a finite subset $\{ U^\gamma y_i \cap A : i = 1,2, \dots ,n\}$ such that

$$A = \bigcup_{i=1}^{n} cl_\gamma(U^\gamma y_i \cap A) = \bigcup_{i=1}^{n}(U^\gamma y_i \cap A).$$

Now corresponding to each $y_i$ , let $U^\gamma xy_i$ be the $\gamma$-open set containing x, then $U^\gamma_x = \cap U^\gamma xy_i$ is $\gamma$-open containing x, since $\gamma$ is regular. Also

$$U^\gamma_x \cap A = U^\gamma_x \cap (\bigcup_{i=1}^{n}(U^\gamma y_i \cap A)) \subseteq U^\gamma_x \cap (\bigcup_{i=1}^{n} U^\gamma y_i)$$

$$\subseteq \bigcup_{i=1}^{n}(U^\gamma_x \cap U^\gamma y_i) = \emptyset$$

or $U^\gamma_x \cap A = \emptyset$. Hence $U^\gamma_x \subseteq X - A$ implies $x \in int_\gamma(X - A)$. Consequently, $X - A = int_\gamma(X - A)$. That is, X −A is $\gamma$-open. So, A is $\gamma$-closed. This completes the proof.

**Definition 3**. Let X be a space and $A \subseteq X$. Then the class of $\gamma$-open sets in A is defined in a natural way as :

$$\tau\gamma_A = \{ A \cap O : O \in \tau_\gamma \},$$

where $\tau_\gamma$ is the class of $\gamma$-open sets of X. That is, G is $\gamma$-open in A iff $G = A \cap O$, where O is a $\gamma$-open set in X.

## 2. $\gamma^s$-regular spaces

**Definition 4**. A space X is said to be $\gamma^s$-regular space, if for any closed set A and $x \notin A$, there exist open sets U , V such that $x \in U$, $A \subseteq V$ and $U^\gamma \cap V^\gamma = \emptyset$.

**Example**. Let X= {a,b,c}, τ ={ ∅, X, {a}, {b,c}}.
For b∈X, define an operation $\gamma : \tau \to P(X)$ by

$$\gamma(A) = \begin{cases} A, & \text{if } b \in A \\ cl(A), & \text{if } b \notin A \end{cases}$$

Then easy calculations show that X is a $\gamma^s$-regular space.

**Theorem 3**. Every subspace of $\gamma^s$-regular space X is $\gamma^s$-regular, where γ is regular.

**Proof** . Let Y be a subspace of a $\gamma^s$-regular space X. Suppose A is γ-closed set in Y and y∈Y such that $y \notin A$. Then $A = B \cap Y$, where B is γ-closed in X. Then $y \notin B$. Since X is $\gamma^s$-regular, there exist open sets U , V in X such that $y \in U$, $B \subseteq V$ and $U^\gamma \cap V^\gamma = \emptyset$. Then $U \cap Y$ and $V \cap Y$ are open sets in Y containing y and A respectively, also

$$\begin{aligned}(U \cap Y)^\gamma \cap (V \cap Y)^\gamma &\subseteq ( U^\gamma \cap Y^\gamma ) \cap ( V^\gamma \cap Y^\gamma ) \quad (\gamma \text{ is regular}) \\ &= ( U^\gamma \cap V^\gamma ) \cap Y^\gamma \\ &= \emptyset \cap Y^\gamma = \emptyset.\end{aligned}$$

This completes the proof.

## 3. $\gamma^s$-normal spaces

**Definition 5**. A space X is said to be $\gamma^s$ -normal space, if for any disjoint closed sets A , B of X, there exist open sets U, V such that $A \subseteq U$, $B \subseteq V$ and $U^\gamma \cap V^\gamma = \emptyset$.

**Example** . Let X= {a,b,c,d},
τ ={ ∅, X, {a}, {b}, {a,b}, {b,d}, {a,b,d}, {b,c}, {b,c,d}, {a,b,c}}.

For b∈X, define an operation $\gamma : \tau \to P(X)$ by

$$\gamma(A) = \begin{cases} cl(A), & \text{if } b \in A \\ clintcl(A), & \text{if } b \notin A \end{cases}$$

Then X is $\gamma^s$-normal.

Next, we characterize $\gamma^s$ -normal space as :

**Theorem 4**. A space X is $\gamma^s$ -normal if for any closed set A and open set U containing A, there is an open set V containing A such that

$$A \subseteq V \subseteq cl_\gamma(V^\gamma) \subseteq U^\gamma,$$

where γ is γ-open and strictly regular.

**Proof** . Let A , B be disjoint closed sets in X. Then $A \subseteq X-B$, where X−B is open in X. By hypothesis, there is a open set V such that

$$cl_\gamma(V^\gamma) \subseteq ( X- B)^\gamma \quad \ldots \quad (1)$$

(1) gives $B^\gamma \subseteq (X-cl_\gamma(V))^\gamma$ and $V \cap (X- cl_\gamma(V^\gamma)) = \emptyset$. Consequently, $A \subseteq V$, $B \subseteq X- cl_\gamma(V^\gamma)$ and
$V^\gamma \cap ((X- cl_\gamma(V^\gamma)))^\gamma = \emptyset$.

This proves that X is γ-normal. This completes the proof.

**Theorem 5**. A $\gamma^s$ -normal γ-$T_1$ space is $\gamma^s$ –regular, where γ is strictly regular.

**Proof** . Suppose A is a closed set and $x \notin A$. Since X is a γ-$T_1$ space, therefore by Proposition 4.9 ( Ogata, 1991) each {x} is γ-closed in X. Since X is $\gamma^s$-normal, therefore there exist open sets U , V such that $\{x\} \subseteq U$, $A \subseteq V$ and $U \cap V = \emptyset$, or $x \in U$, $A \subseteq V$ and $U \cap V = \emptyset$ implies that $U^\gamma \cap V^\gamma = \emptyset$, since γ is strictly regular. Thus X is $\gamma^s$ -regular.This completes the proof.

**Theorem 6**. A closed subspace of a $\gamma^s$ -normal space X is $\gamma^s$ -normal, where γ is regular.

**Proof** . Let A be a closed subspace of $\gamma^s$ -normal space X. Let $A_1$, $A_2$ be disjoint closed sets of A. Then there are closed sets $B_1$, $B_2$ in X such that $A_1 = B_1 \cap A$ , $A_2 = B_2 \cap A$. Since A is closed in X, therefore $A_1$, $A_2$ are closed in X. Since X is $\gamma^s$-normal, there exist open sets $U_1$, $U_2$ in X such that $A_1 \subseteq U_1$, $A_2 \subseteq U_2$ and $U^\gamma_1 \cap U^\gamma_2 = \emptyset$. But then $A_1 \subseteq A \cap U_1$, $A_2 \subseteq A \cap U_2$ , Where $A \cap U_1$, $A \cap U_2$ are open in A and

$$\begin{aligned}(A \cap U_1)^\gamma \cap (A \cap U_2)^\gamma &\subseteq (A^\gamma \cap U^\gamma_1 ) \cap (A^\gamma \cap U^\gamma_2 ) \quad (\text{since } \gamma \text{ is regular}) \\ &= A^\gamma \cap ( U^\gamma_1 \cap U^\gamma_2 ) \\ &= A^\gamma \cap \emptyset = \emptyset.\end{aligned}$$

This proves that A is $\gamma^s$ -normal. Hence the proof.

**Theorem 7**. Every $\gamma_0$-compact and γ-$T_2$ space is $\gamma^s$-normal, where γ is regular and γ-open.

**Proof**. Let X be $\gamma_0$-compact and γ-$T_2$ space and $C_1$ , $C_2$ be ant two disjoint γ-closed subsets of X. Then being γ-closed subset of $\gamma_0$-compact space, $C_1$ is $\gamma_0$-compact. By Theorem 1, for $\gamma_0$-compact $C_2$ and $x \notin C_2$, there are open sets $U_x$ , $V_x$ such that

$$x \in U^\gamma_x , C_2 \subseteq V^\gamma_x \text{ and } U^\gamma_x \cap V^\gamma_x = \emptyset. \quad \ldots \quad (2)$$

Let the set $\{ U^\gamma_x : x \in C \}$ be a cover of $C_1$ by γ-open and γ-closed sets of $C_1$ . Since $C_1$ is $\gamma_0$-compact, so there are finite number of elements $x_1, x_2, \ldots , x_n$ such that

$$C_1 \subseteq \bigcup_{i=1}^{n} cl_\gamma (U^\gamma x_i) = \bigcup_{i=1}^{n} (U^\gamma x_i).$$

Let $U = \bigcup_{i=1}^{n} (U^\gamma x_i)$ , $V = \bigcap_{i=1}^{n} (V^\gamma x_i)$.

Then $C_1 \subseteq U$, $C_2 \subseteq V$ and

$$(U \cap V)^\gamma = ((\bigcup_{i=1}^{n} (U^\gamma x_i)) \cap (\bigcap_{i=1}^{n} (V^\gamma x_i)))^\gamma$$

$$= (\varnothing)^\gamma = \varnothing.$$

Hence X is $\gamma^s$-normal. This completes the proof.

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