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# A REVIEW ON THE BOOK BY KHMELNIK: NAVIER-STOKES EQUATIONS ON THE EXISTENCE AND THE SEARCH METHOD FOR GLOBAL SOLUTIONS

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## ABSTRACT

In 2010, Khmelnik has developed his theoretical method applicable for resolving problems existing in mechanics, electrodynamics, electrical engineering, hydrodynamics. For hydrodynamics, this method allowed for Khmelnik to resolve the Navier-Stokes equations. However, this work by Khmelnik is not widely known. As a result, there are still both analytical and numerical attempts to find a suitable method of resolving of the Navier-Stokes equations. The work by Khmelnik provides the obtained results in both analytical forms and color graphical three-dimensional illustrations. Therefore, this review has the purpose to briefly acquaint the reader with the resolving method that was successfully applied to the Navier-Stokes equations and the other problems in physics.

**Keywords:** Navier-Stokes equations, hydrodynamics, lagrangian formalism, compressible and incompressible fluids, turbulent flow.

## INTRODUCTION

It is mentioned in (Bratsun and Vyatkin, 2019) that in hydrodynamics at pre-computer times, to find exact solutions for some cases described by hydrodynamic equations was irreplaceable event. Indeed, to obtain exact solutions means to have the key tools for fluid motions' obtaining information. Let's mention here some classical examples: the Taylor flow confined between two rotating coaxial cylinders (Taylor, 1923), the Poiseuille flow in a pipe under applied pressure drop (Poiseuille, 1840), and the Couette flow between two surfaces, one of which is moving tangentially relative to the other (Couette, 1890).

However, there are problems in hydrodynamics that still are unresolved during the last centuries. One of such problems is the Navier-Stokes equations. The importance of resolving of the Navier-Stokes equations consists in their multi-promising applications. As a result, the Clay Mathematics Institute (CMI), see here http://www.claymath.org/millennium-problems/navier%E2%80%93stokes-equation, has included this problem in the list of the seven most important mathematical problems to be resolved in this millennium, see also in (Ladyzhenskaya, 2003). Khmelnik from Israel has stated that he has resolved the Navier-Stokes equations and he has published his method of resolving of the Navier-Stokes equations in 2010 in both Russian (Khmelnik, 2010a) and English (Khmelnik, 2010b). Several years ago, Dr. Khmelnik has contacted to the Clay Mathematics Institute, sending his book with his method of resolving of the Navier-Stokes equations and claiming that he has resolved this problem. The Clay Mathematics Institute has answered that somebody (but not Dr. Khmelnik himself) should introduce the work by Khmelnik to the CMI in or-der that the CMI can treat his method.

This review has the purpose to quaint the reader with already the sixth edition (Khmelnik (2021)) of his results initially published in 2010 (Khmelnik, 2010a, 2010b). This last sixth edition of the book represents the three-books-in-one edition because two books by Khmelnik (2018a, 2018b) with the computer programs in the MATLAB codes and their explanations, respectively, were added as Appendices 8 and 9 to the fifth edition (Khmelnik, 2018c) of his book to compose the sixth edition. So, the book by Khmelnik (2021) is supplemented by the corresponding computer program codes of the MATLAB platform, namely by functions that realize the calculation method and the computer program test codes.

In his book, Khmelnik (2010b, 2018, 2021) starts with the Lagrangian formalism in Chapter 1 and introduces the Energian that can be understood as an extended or modified Lagrangian. It is well-known that the Lagrangian formalism is used for many problems of physics, for instance, it was recently applied for antimatter gravity study (Jentschura, 2020). So, the main formulas in the book Khmelnik (2021) are written down

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in Chapters 1 and 2 as well as in the appendices from 1 to 7. The obtained method by Khmelnik can be applied to many problems. Today the reader can find a lot of literature where the Navier-Stokes equations are used. For instance, within the last several years it is possible to find ~ 100 published papers among only the MDPI Journals such as Mathematics, Fluids, and Mathematical and Computational Applications. This review has no objective to review all of them. However, it is possible to mention some of them to demonstrate different problems where the Navier-Stokes equations can be applied.

Ersoy *et al*. (2021) have derived the model via asymptotic reduction from the two-dimensional Navier-Stokes equations under the shallow water assumption, with boundary conditions including recharge via ground infiltration and runoff. Ersayın and Selimefendigil (2013) have numerically investigated the effects of various parameters such as nanoparticle volume fraction, pulsating frequency, plate velocity, Reynolds number on the heat transfer characteristics, where the Navier-Stokes equations and energy equations are solved with a commercial finite volume based code. Moschandreou (2018) has proposed a solution to the three-dimensional Navier-Stokes equations in the cylindrical coordinates coupled to the continuity and level set convection equation. Akkari *et al*. (2019) have considered the problem of constructing a time stable reduced order model of the three-dimensional Navier-Stokes equations in the incompressible and turbulent case. Semenov (2014) has discussed the exact estimates for solutions in the Cauchy problem for the Navier-Stokes equations and Euler equations. Metivet *et al*. (2018) have presented the simulation of multifluid flows based on the implicit level-set representation of interfaces and on an efficient solving strategy of the Navier-Stokes equations. In the study by Kang and You (2021), a cell-centered finite-volume method for compressible Naiver-Stokes equations was developed. Saito (2021) has shown time-decay estimates of solutions to linearized two-phase Navier-Stokes equations with surface tension and gravity. Galdi (2021) has provided conditions for the occurrence of time-periodic Hopf bifurcation for the coupled system constituted by a rigid sphere moving under gravity in a Navier-Stokes liquid. The paper by Azlan *et al*. (2021) has studied the linearized problem for the compressible Navier-Stokes equation around space-time periodic state. Kubo and Shibata (2021) have treated the problem formulated mathematically by the Navier-Stokes equations in a time-dependent domain separated by an interface, where one part of the domain is occupied by a compressible viscous fluid and another part by an incompressible one. Aksenov and Polyanin (2021) have described a number of simple methods for constructing exact solutions of nonlinear partial differential equations, including the Navier-Stokes equations, that involve a relatively small amount of intermediate calculations. Bourantas (2021) has studied a model consisting of the Navier-Stokes equations expressed in the velocity-vorticity variables including the energy and microrotation transport equation. For the Navier-Stokes equations, Yakhlef and Murea (2021) have used the fictitious domain method with penalization. Mimeau and Mortazavi (2021) have reviewed the vortex methods belonging to Lagrangian approaches and allowing one to solve the incompressible Navier-Stokes equations in their velocity-vorticity formulation. The work by Sarthou *et al*. (2020) has studied the interactions between fictitious-domain methods on structured grids and velocity–pressure coupling for the resolution of the Navier-Stokes equations. Murea (2019) has used the updated Lagrangian framework for the linear elasticity equation modeling the structure and the Navier-Stokes equations governing the fluid. In (Jabbari *et al*., 2019), a first-order projection method was used to numerically solve the Navier-Stokes equations for a time-dependent incompressible fluid inside a 3D lid-driven cavity.

Also, there are a lot of publications where the method called the Reynolds-averaged Navier-Stokes equations (or RANS equations) is used. among the aforementioned 100 articles on the Navier-Stokes equations, at least half of them refers to this specified method. For instance, Huilier (2021) has stated that instead of Reynold-Averaged Navier-Stokes (RANS)-based studies, the computer evolution and performance allowed development of large eddy simulation (LES) and direct numerical simulation (DNS) of turbulence coupled to Generalized Langevin Models. A dual approach is applied in (Wenig *et al*., 2021), in which the LES is used as reference for the unsteady RANS computations. Mai and Ryu (2021) have used the RANS equation coupled with the turbulence model to solve the problem of high-speed and high-pressure compressible flow through the gas-turbine model. In (Lluesma-Rodríguez *et al*., 2021), it was designed a numerical method to solve the time-dependent incompressible 3D Navier-Stokes equations in turbulent thermal channel flows. Also, Lluesma-Rodríguez *et al*. (2021) state that it is well-known that we still even lack an existence and uniqueness theorem about the solution of the governing equations of turbulent flows, the Navier-Stokes equations.

In Chapter 4 of the book by Khmelnik (2021), it is discussed that strictly speaking the treated RANS equations cannot be called Navier-Stokes equations and the considered method of constructing the functional is not applicable to them. In Chapter 10 of the book, the other method of calculation of turbulent flows is demonstrated. For turbulent flows, some equations similar to the Navier-Stokes equations can be obtained. Next, a functional can be constructed for the obtained turbulent flow equations and these equations represent the necessary conditions for the existence of the extremum for this constructed functional. The turbulent flow equations are equations (10.8.1) and (10.8.1) in Subsection 10.8 of Chapter 10 in the book by Khmelnik (2021). This functional is not convex. Therefore, there may be more than one solution for these turbulent flow equations, read the context after equation (10.8.9) in Subsection 10.8. However, there are flows with weak turbulences, which can be represented by the sum of a high-speed laminar flow and a turbulent flow with relatively low speeds. Such flows are described by two independent set of equations and there are unique solutions for each of these sets, see the algorithm at the end of Subsection 10.8.

The author of this review is familiar with many published papers and books by Solomon Itskovich Khmelnik from Israel, including his work on the Navier-Stokes equations (Khmelnik, 2021)and has contacts with him already during the last two years. The author has found that Khmelnik has developed a variational principle that can be applied to dissipative systems. This review has no aim to rewrite all the results obtained by Khmelnik (the reader must read his book (Khmelnik, 2021)) but only concisely acquaints the reader with the main results obtained by Khmelnik and states that namely the Navier-Stokes equations were resolved by Khmelnik in 2010, according to his first work published in both Russian (Khmelnik, 2010a) and English (Khmelnik, 2010b).

**The concise description of the method by Khmelnik (2021)**

If in the existing principle the integrand is the difference between the kinetic energy and the potential energy, then in the proposed functional the integrand is the difference (ΔH) between the kinetic energy and the sum of the potential and thermal energies. In the absence of a dissipation process, such an expression turns into Lagrangian. When constructing a functional, each desired function splits into two independent functions, and the functional contains pairs of such split functions. The integrand itself depends on pairs of independent split functions. The optimum of a functional is the saddle point where one group of split functions minimizes the functional and the other maximizes it. The sum of the pair of optimal values of these split functions gives the desired function. The variational principle in this case states that the extremals of this functional (defining the saddle point) are real dynamic variables that can be realized in reality. Khmelnik calls it the principle of general action.

This principle allows one to design a functional for various physical systems and, most importantly, for dissipative systems. This functional has a global saddle point. Therefore, to calculate physical systems with this functional, one can apply the method of gradient descent to the saddle point. The solution always exists because the global extremum exists. The initial step in constructing this functional is as follows: for some physical system, the energy conservation equation or the power balance equation is written, taking into account both energy losses (for instance, friction or heating) and the flow of energy into and out of the system.

The proposed variational principle allows one to construct a functional for various physical systems with a single saddle point of the optimum. Khmelnik also proposed a computational method for moving to the saddle point, which allows one to calculate the extremals of this functional. Therefore, the real equations of motion for some treated system can be determined. Thus, the new formalism is not only a universal method for deriving physical equations from a certain principle but also a way to calculate these equations.

Khmelnik has applied this principle to various physical problems. First of all, the proposed formalism is applicable to electrical circuits. It should be noted that for DC circuits (where there are only resistances) the solution was found by Maxwell, who showed that the currents are distributed in such a way that they minimize heat losses. However, for AC electrical circuits, a principle similar to the principle of least action has not yet been found in spite of the fact that there were many attempts. These searches are understandable because the absence of the principle of extremality for electrical circuits seems strange. The proposed variational principle for electrical circuits is such that it follows from the Kirchhoff equations for AC electrical circuits. This means that there is a definite integral of the split functions of charges, which are functions of time. This integral has a single optimum point and its extremals coincide with the Kirchhoff equation. Thus, in the electrical circuit, the principle of the extremum of the aforementioned value of ΔH is objectively observed.

This functional can be also found for electric power systems, which are nonlinear electrical circuits. It is shown that the resulting functional is optimized when the stationary value of the integrand function is the equation of the power system mode. Khmelnik has also demonstrated that the solution of a set of linear algebraic equations is also reduced to the calculation of an electric circuit of sinusoidal currents by the proposed method. It was shown that the solution of sets of partial differential equations also reduces to the search for the optimum of a certain functional. For instance, the equations of Poisson, Helmholtz for homogeneous and inhomogeneous media, the telegraph equation, and many others can be resolved by the proposed method. In addition, this principle is generalized to electromechanical systems. For some treated electromechanical system, the formed functional contains thermal, mechanical, electrical, and magnetic energies, functions depending on the configuration of the treated system, and functions describing perturbations' effects both electrical and mechanical. The same method can be readily applied to Maxwell's equations, which involve currents in a medium with a certain electrical conductivity. Therefore, there is a heat loss, i.e. energy dissipation. This means that in addition to electric and magnetic energies, the functional for the principle of minimum action must include thermal energy. Hence, the Lagrange formalism for Maxwell's equations is not applicable in this case.

It should be mentioned that the calculus of variations has one drawback because it is assumed that the functions are continuous. In practice, discontinuous functions such as the Dirac function and step functions are often found. For optimization problems with such functions there is a Pontryagin maximum method. Khmelnik has demonstrated that the Pontryagin maximum method can be combined with the method of gradient motion to the saddle point. As a result, an algorithm for resolving a set of Maxwell's differential equations with discontinuous perturbations is formulated. All calculations in this algorithm consist of operations with coefficients of polynomial functions. The result is also given in the form of polynomial functions, i.e. an analytical solution is obtained.

The most interesting thing is how Khmelnik has applied this principle in hydrodynamics, namely to the problem of proving the uniqueness of the solution of the Navier-Stokes equations that has not been resolved for the last couple of centuries. As an outstanding researcher, Khmelnik works alone already for a long time and therefore, his results are not widely known, although the solution to this problem was first published in 2010 in Russian (Khmelnik, 2010a) and then in 2010 in English (Khmelnik, 2010b). In this regard, it is necessary to briefly describe below the resolving method found by Khmelnik. This method is suitable for the three-dimensional case. It worce here noting that in the two-dimensional case, the the solution metod has been knownfor a long time (Ladyzhenskaya, 1969), also for the more difficult case of the Eulerequations. However, this gives no hint about the three-dimensional case because the maindifficulties are absent in the two-dimensional case.

**The resolving method by Khmelnik (2021) for the Navier-Stokes equations**

First of all, Khmelnik offers that the following functional (certain integral) can be called some full action: $\Phi(q) = \int_{t_1}^{t_2} \Re(q)dt$, where $\Re(q) = (K(q) - P(q) - Q(q))$ is called the Energian by Khmelnik (by analogy with the Lagrangian.) $K(q)$, $P(q)$, $Q(q)$ are the kinetic, potential, and thermal energies, respectively. $q$ is a vector of generalized coordinates, dynamic variables that depend on time $t$. Here, the integral is taken at a certain time interval $t_1 \le t \le t_2$. It should be noted here that when $Q$ = 0, the functional $\Phi(q)$ turns into some action.

Next, it is natural to consider the quasi-extremal of the full action, which has the following form:

$$\frac{\partial(K-P)}{\partial q} - \frac{d}{dt}\left(\frac{\partial(K-P)}{\partial q'}\right) - \frac{\partial Q}{\partial q} = 0 \qquad (1)$$

The functional $\Phi(q)$ definitely takes an extreme value on quasi-extremals. The principle of extremum of full action states that the quasi-extremals of this functional are equations of real dynamic processes. It is possible to determine the extreme value of the functional $\Phi(q)$. For this purpose, it is possible to split the function $q(t)$ into two independent functions $x(t)$ and $y(t)$, and to correspondingly treat the following functional instead of the functional $\Phi(q)$: $\Phi_2(x,y) = \int_{t_1}^{t_2} \Re_2(x,y)dt$ called the “split” full action. The function $\Re_2(x,y)$ is called the “split” Energian. This functional is minimized by the function $x(t)$ for a fixed function $y(t)$ and maximized by the function $y(t)$ for a fixed function $x(t)$. The minimum and maximum are the only ones.

Thus, the extremum of the functional $\Phi_2(x,y)$ is a saddle line, where one group of functions ($x_0$) minimizes the functional, and the other ($y_0$) maximizes it. The sum of the pair of optimal values of the split functions gives the desired function $q = x_0 + y_0$, which satisfies the aforementioned equation of quasi-extremal (1). In other words, the quasi-extremal of the functional $\Phi(q)$ is the sum of the extremals ($x_0$, $y_0$) of the functional $\Phi_2(x,y)$, which determine the saddle point of this functional. It is important to note that this point is the only extreme point because there are no other saddle points and no other minimum or maximum points. This is the meaning of the expression "extreme value on quasi-extremals". The statement is that in every field of physics, one can find a correspondence between the complete action and the split complete action, and thus prove that the complete action takes on a global extreme value on quasi-extremals.

In order to show the validity of this statement for hydrodynamics, Khmelnik uses the well-known work of Umov, who considered for a liquid the condition of the balance of specific (by volume) capacities in the liquid flow for both incompressible and compressible liquids. From this condition, after long transformations, it is possible to find the quasi-extremal equation. Next, Khmelnik constructs a computational algorithm. Stationary and dynamic problems are also considered, and the proposed algorithm is implemented in MATLAB program codes. Examples of flows are also shown graphically and corresponding MATLAB program codes are also considered in the book.

Let's also shortly introduce the ten chapters of the book. Chapter 1 acquires the reader with the principle of extremum of full action. Chapter 2 this principle for viscous incompressible fluids. Chapter 3 briefly discusses the computational algorithms applied. Chapter 4 concisely discusses the method called the Raynolds-average Navier-Stokes equations (RANS equations) stating that strictly speaking the RANS equations cannot be called Navier-Stokes equations and the method of constructing the functional is not applicable to the RANS equations. Chapters 5 and 6 treat the stationary and dynamic problems, respectively. Chapters 7 and 8 provide the following examples: Computations for a mixer and flow in a pipe, respectively. In Subsections 7.2 and 7.3 of Chapter 7, the polar and Cartesian coordinates are also treated. In these chapters, the results are also shown graphically in eleven and thirteen figures, respectively, some of them are three-dimensional plots. Chapter 9 describes the principle of extremum of full action for viscous compressible fluids. Chapter 10 represents the emergence mechanism and calculation method of turbulent flows. In this chapter, the results were also graphically shown in eleven figures, some of them are three-dimensional plots. In addition, this book also contains nine appendices, of which the first seven contain a lot of complicated formulas used by Khmelnik in his work. Appendices 8 and 9 contain the relevant computer programs in the MATLAB codes and their explanations, respectively.

## CONCLUSION

It is suggested that the reader pay attention to the book on the obtained method of resolving of the Navier-Stokes equations. Indeed, his work has many practical applications.

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