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Short Communication

A NEW APPROACH TO STATISTICAL INFERENCE FOR EXPONENTIAL DISTRIBUTION BASED ON RECORD VALUES

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ABSTRACT

In this paper, we use the upper record range statistic to draw inferences from the scale parameter of the exponential distribution. These inferences are point estimation and interval estimation. We obtained MRE estimations under three different loss functions: Quadratic, Squared error and Absolute error loss function. Moreover, we derive the shortest interval and interval estimation with equal tails. Finally, we present some practical examples and simulations by using the method of inverse distribution transformation.

Keywords: Exponential distribution, scale parameter, record values, upper record range, estimation based on upper record range.

1-INTRODUCTION

Let $X_1, X_2, X_3, \ldots$ be a sequence of independent and identically distributed (iid) random variable with cumulative distribution function (cdf) F(x) and probability density function (pdf) f(x). For $n \geq 1$ define
$T(1) = 1, T(n+1) = min\{j: X_j > X_{T(n)}\}$.

The sequence $\{X_{T(n)}\}_{n=1}^{\infty}$ is known as upper record values and the sequence $\{T(n)\}_{n=1}^{\infty}$ is known as record times sequence (Arnold *et al*., 1998). At first Chandler (1952) introduced the concepts of record values, times and related statistics theoretically. Interested reader may refer to Arnold *et al*., 1998; Nagaraja, 1988; Nevzorov 1946 for basic concepts of this subject. Some inferential studies based on record values and record times have been done by Ahsanullah (1990), Balakrishnan *et al*. (1995), Feuerverger and Hall (1998). Consider the one-parameter Exponential distribution with scale parameter δ and pdf

$$f(x;\delta) = \frac{1}{\delta} e^{-\frac{x}{\delta}}, x \geq 0, \delta > 0 . \quad (1)$$

and cdf

$$F(x;\delta) = 1 - e^{-\frac{x}{\delta}}, x \geq 0, \delta > 0 . \quad (2)$$

The exponential distribution occurs naturally when describing the lengths of the inter-arrival times in a homogeneous Poisson process. Exponential variables can also be used to model situations where certain events occur with a constant probability per unit length, such as the distance between mutations on a DNA strand, or between road kills on a given road. In queuing theory, the service times of agents in a system (e.g. how long it takes for a bank teller etc to serve a customer) are often modeled as exponentially distributed variables. Reliability theory and reliability engineering also make extensive use of the exponential distribution. In physics, if you observe a gas at a fixed temperature and pressure in a uniform gravitational field, the heights of the various molecules also follow an approximate exponential distribution. In hydrology, the exponential distribution is used to analyze extreme values of such variables as monthly and annual maximum values of daily rainfall and river discharge volumes. Balakrishnan *et al*. (1995) have established some recurrence relations for single and product moments of record values from exponential distribution based on record values. Ahmadi *et al*. (2005) have obtained estimation and prediction based on k-record values for two-parameter exponential distribution. Ahsanullah and Kirmani (1991) have obtained some characterizations of the exponential distribution based on lower record values. Finding different methods for statistical inference based on records is the purpose of this article. In the situation that we have only the smallest and the largest data, the statistics based on $X_{T(1)}$ and $X_{T(n)}$ will play an important role in the statistical inferences. This occurs in many real situations such as stock exchange. Consider a statistician who wants to make the statistical inferences about the prices of stocks and shares in a stock market. Usually the middle prices are not recorded and we have only the largest and the base prices. Pharmacy is another instance. For confirming the effectiveness of drugs and poisons the upper and lower levels of effectiveness is used. Because the decisions are made based on the largest and the smallest values, they are of importance. When we want to make inference based on the largest record and the smallest record values, one of the best choices is $R_{U,R}$

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which is defined by $R_{U,R} = X_{T(n)} - X_{T(1)}$. In this article we try to make inference based on this statistic therefore in section 2, we estimate the scale parameter δ using the classical methods consisting MLE and MME, MRE estimation. In section 3, we determine shortest interval and interval with equal tail probabilities based on $R_{U,R}$ also in section 4; we present some practical examples and simulations.

2-UPPER RECORD RANGE AND POINT ESTIMATIONS

Suppose that in a random sample from a population with probability density function f(x), the record values are $X_{T(1)}, X_{T(2)}, \dots, X_{T(n)}$. The joint pdf $f(x_{T(1)}, x_{T(2)}, \dots, x_{T(n)})$ is (see [1])

$$f(x_{T(1)}, x_{T(2)}, \dots, x_{T(n)}; \delta) = f(x_{T(n)}; \delta) \prod_{i=1}^{n} h(x_{T(i)}; \delta), \quad (3)$$

where $h(x_{T(i)}; \delta) = \frac{f(x_{T(n)};\delta)}{1-F(x_{T(n)};\delta)}$.

Therefor from (1) and (2), (3) we have

$$f(x_{T(1)}, x_{T(2)}, \dots, x_{T(n)}; \delta) = \frac{1}{\delta^n} \exp\left(-\frac{x_{T(n)}}{\delta}\right),$$

where $x_{T(1)} < x_{T(2)} < \dots < x_{T(n)}$.

Integrating out $X_{T(2)} \dots X_{T(n-1)}$, we get the join pdf $f(x_{T(1)}, x_{T(n)})$ as

$$f(x_{T(1)}, x_{T(n)}) = \frac{1}{(n-2)!\delta^n} (x_{T(n)} - x_{T(1)})^{n-2} \exp\left(-\frac{x_{T(n)}}{\delta}\right),$$

where

$0 < x_{T(1)} < x_{T(n)} < \infty$.

Using the transformations $R_{U,R} = X_{T(n)} - X_{T(1)}$, $U = X_{T(n)}$ and integrating out U, we obtain pdf $f_{R_{U,R}}(r)$ of $R_{U,R}$ as

$$f_{R_{U,R}}(r) = \frac{r^{n-2} \exp\left(-\frac{r}{\delta}\right)}{(n-2)!\delta^{n-1}}, r > 0 \quad (4)$$

This means that $R_{U,R} = X_{T(n)} - X_{T(1)}$ is distributed as $amma(n-1, \delta)$.

A. The Method of Maximum Likelihood Estimation (MLE) Based on Upper Record Range Statistic

From (4) log likelihood function is,

$$L(\delta; r) = \log\left(f_{R_{U,R}}(r; \delta)\right) = (n-2)\log(r) - \log(n-2)! - (n-1)\log(\delta) - \frac{r}{\delta} \quad (5)$$

The MLE of δ can be obtained by solving the following likelihood equation

$$\frac{dL}{d\delta} = 0. \quad (6)$$

By solving equation (6) the MLE estimation based on upper record range for the parameter δ can be obtained as

$$\hat{\delta}_{MBURR} = \frac{X_{T(n)} - X_{T(1)}}{n-1}, \quad (7)$$

Note that

$$E[\hat{\delta}_{MBURR}] = \delta, Var(\hat{\delta}_{MBURR}) = \frac{\delta^2}{n-1}.$$

By considering the Factorization theorem and rewriting (4) as below we find that $R_{U,R}$ is the sufficient and complete statistic for the parameter δ.

$$f_{R_{U,R}}(r; \delta) = \frac{r^{n-2} \exp\left(-\frac{r}{\delta}\right)}{(n-2)!\delta^{n-1}} = h(r) g(r, \delta). \quad (8)$$

Therefore from (7) and (8), $\hat{\delta}_{MBURR}$, is equal to $\hat{\delta}_{UMVUE}$.

B. THE METHOD OF MOMENT ESTIMATION

Let $R_{U,Ri}$ be the iid sample from Upper Record Range statistic. This occurs in the situation that we have m random samples by size n from the pdf $f(x)$. In fact this occurs because we have an upper record range per sample. In this situation the MM estimation is obtained by solving the moment equations

$$E[R_{U,R}^K] = \frac{1}{m} \sum_{i=1}^{m} r_i^K. \quad (9)$$

From (4) and putting k=1 in (9), the moment equation can be derived as

$$(n-1)\delta = \frac{1}{m} \sum_{i=1}^{m} r_i.$$

Consequently the MM estimation is given by

$$\hat{\delta}_{MMEBURR} = \frac{1}{m(n-1)} \sum_{i=1}^{m} R_{U,Ri}$$

Note that

$$E[\hat{\delta}_{MMEBURR}] = \delta, Var(\hat{\delta}_{MMEBURR}) = \frac{\delta^2}{m(n-1)}$$

C. MINIMUM RISK EQUIVARIANT (MRE) ESTIMATIONS

In this section we consider three kinds of loss functions to obtain the MRE estimation based on $R_{U,R}$ for the scale parameter δ. The loss functions are i-Quadratic loss ii-Squared error loss iii-Absolute error loss function. Let G be a group of transformations in the form

$$G = \{g_a : g_a(x) = ax, a > 0\}.$$

Then the exponential distribution (1) and the loss functions (i) and (ii), (iii) are invariant under G. By this introduction we try to obtain the MRE estimations based on upper record range in three subsections.

1. MINIMUM RISK EQUIVARIANT (MRE) ESTIMATION BASED ON UPPER RECORD RANGE UNDER THE QUADRATIC LOSS FUNCTION

By considering quadratic loss function (i) the minimum risk equivariant estimation (MRE) of δ is given by Lehmann and Casella (1998)

$$\delta^*(X) = \frac{\delta_0(X)}{\omega^*(X)}, \quad (10)$$

where δ_0 is any scale equivariant estimator of δ and $\omega(Z) = \omega^*(Z)$ minimizes

$$E_{\delta=1}\left[\gamma\left\{\frac{\delta_0(X)}{\omega^*(X)}\right\}\Big|Z\right], \quad (11)$$

where $Z = \frac{R_{U,R}}{|R_{U,R}|}$ and γ is an invariant loss function. By considering quadratic loss function (i) we have

$$\omega^*(Z) = \frac{E_{\delta=1}[\delta_0^2(X)|Z]}{E_{\delta=1}[\delta_0(X)|Z]}.$$

Because of equvariance of the MLE based on $R_{U,R}$ ($\hat{\delta}_{MBURR}$), we assume $\delta_0 = \hat{\delta}_{MBURR}$. Since Z is ancillary, by Basu's theorem, δ_0 is independent of Z, and hence

$$\omega^*(Z) = \frac{E_{\delta=1}[\delta_0^2(X)]}{E_{\delta=1}[\delta_0(X)]} = \frac{n}{n-1}. \quad (12)$$

From (10), (12) and by considering $\delta_0 = \hat{\delta}_{MBURR}$ the MRE estimation based on $R_{U,R}$ is

$$\delta^*_{MRE,1} = \frac{\frac{X_{T(n)}-X_{T(1)}}{n-1}}{\frac{n}{n-1}} = \frac{X_{T(n)}-X_{T(1)}}{n}.$$

Note that

$$E[\delta^*_{MRE,1}] = \frac{n-1}{n}\delta \quad , \quad Var(\delta^*_{MRE,1}) = \frac{n-1}{n^2}\delta^2.$$

Therefore the MRE estimation based on $R_{U,R}$ under quadratic loss function is asymptotically unbiased. It also converges in probability to the parameter δ.

II. MINIMUM RISK EQUIVARIANT (MRE) ESTIMATION BASED ON UPPER RECORD RANGE UNDER THE SQUARED ERROR LOSS FUNCTION

Following the last subsection, and considering (10), (11) with Squared error loss function we can obtain the $\omega^*(X)$ as

$$\omega^*(Z) = E_{\delta=1}[\delta_0|Z].$$

Because Z is an ancillary statistic it satisfies the conditions of Basu's theorem, so δ_0 is independent from Z and we have

$$\omega^*(Z) = E_{\delta=1}(\delta_0) = \frac{n-1}{n-1} = 1. \quad (13)$$

Consequently from (10) and (13) we can easily obtain the MRE estimation based on $R_{U,R}$ under squared error loss function as

$$\delta^*_{MRE,2} = \frac{\frac{X_{T(n)}-X_{T(1)}}{n-1}}{1} = \frac{X_{T(n)}-X_{T(1)}}{n-1}.$$

As it is seen the obtained MRE in this subsection ($\delta^*_{MRE,2}$) is equal to MLE estimation. Therefor

$$E[\delta^*_{MRE,2}] = \delta \quad , \quad Var(\delta^*_{MRE,2}) = \frac{\delta^2}{n-1}.$$

As we can see, $\delta^*_{MRE,2}$ converges in probability to δ and also has asymptotically unbiased property.

III. MINIMUM RISK EQUIVARIANT (MRE) ESTIMATION BASED ON UPPER RECORD RANGE UNDER THE SQUARED ERROR LOSS FUNCTION

Considering the absolute error loss function (iii) and $\delta_0 = \hat{\delta}_{MBURR}$, because δ_0 is independent from Z (Basu's theorem), ω^* is

$$\omega^*(Z) = Median_{\delta=1}(\delta_0|Z) = Median_{\delta=1}(\delta_0).$$

After some algebraic manipulation $\omega^*(Z)$ is given by

$$\omega^*(Z) = Median_{\delta=1}(\hat{\delta}_{MBURR}) = \frac{\chi^2_{2n-2;0.5}}{2n-2},$$

and consequently by (10), (11), the MRE estimation based on $R_{U,R}$ under absolute error loss function ($\delta^*_{MRE,3}$) can be obtained as

$$\delta^*_{MRE,3} = \frac{\delta_0}{\omega^*(Z)} = \frac{2R_{U,R}}{\chi^2_{2n-2;0.5}}.$$

3-INTERVAL ESTIMATION BASED ON UPPER RECORD RANGE STATISTIC

A. CONFIDENCE INTERVAL WITH EQUAL TAILS

In this section we obtain two-sided confidence interval with equal tail probabilities based on $R_{U,R}$. Adopting the usual approach we consider the pivot quantity (Q) as a function of minimal sufficient statistic (based on $R_{U,R}$). Therefore from (8)

$$Q = \frac{2R_{U,R}}{\delta}.$$

By considering the pdf of $R_{U,R}$ (4) and obtaining the distribution of Q, it is clear that Q is distributed as chi-squared with $2n-2$ degrees of freedom. For obtaining confidence interval with equal tails, a, b must be determined from these equations

$$P(a < Q < b) = \int_a^b f_Q(t)dt = 1-\alpha, \quad (14)$$

and

$$P(Q < a) = \frac{\alpha}{2}, P(Q > b) = \frac{\alpha}{2}.$$

Some algebraic manipulation gives us a, b as

$$a = \chi^2_{2n-2;\frac{\alpha}{2}}, b = \chi^2_{2n-2;1-\frac{\alpha}{2}}$$

Therefore the confidence interval with equal tail probabilities based on $R_{U,R}$ is given by

$$\frac{2R_{U,R}}{\chi^2_{2n-2;1-\frac{\alpha}{2}}} < \delta < \frac{2R_{U,R}}{\chi^2_{2n-2;\frac{\alpha}{2}}}$$

Consequently the length is

$$L = \frac{2R_{U,R}}{\chi^2_{2n-2;\frac{\alpha}{2}}} - \frac{2R_{U,R}}{\chi^2_{2n-2;1-\frac{\alpha}{2}}}.$$

B. THE SHORTEST INTERVAL BASED ON UPPER RECORD RANGE STATISTIC

In this section we want to obtain the shortest confidence interval based on upper record range. Using the obtained pivot quantity in last section we have the general intervals as $\frac{2R_{UR}}{b} < \delta < \frac{2R_{UR}}{a}$.

Consequently the length of these general interval is $L = 2R_{U,R}(\frac{1}{a} - \frac{1}{b})$.

For obtaining the shortest interval we should choose a, b such that minimize the length (15) and satisfy equation (14). Using Lagrange multipliers method

$$\psi(a,b,\lambda) = 2R_{U,R}\left(\frac{1}{a} - \frac{1}{b}\right) + \lambda\left(\int_a^b f_Q(t)dt - (1-\alpha)\right).$$

After derivation by λ, a, b we have

$$\begin{cases} \int_a^b f_Q(t)dt = 1-\alpha \\ \frac{-2R_{UR}}{a^2} - \lambda f(a) = 0 \\ \frac{2R_{UR}}{b^2} + \lambda f(b) = 0 \end{cases} \Rightarrow \begin{cases} \int_a^b f_Q(t)dt = 1-\alpha \\ a^2 f_Q(a) = b^2 f_Q(b) \end{cases} \quad (16)$$

The equations (16) have solved by Tate and Klett in 1959 numerically .

4-SIMULATION AND ILLUSTRATIVE EXAMPLE

To illustrate the estimation techniques developed in this section, we consider simulated data bellow from Exponential distribution with $\delta = 1$ using the transformation: $x_i = \delta Ln(1-u_i)$ where U_i is the uniformly distributed random variable. Suppose that we observed the following simulated data from $Exp(\delta)$:

1.179469281, 0.613291030, 0.976919804, 0.940902389, 3.151606189, 0.454850203,0.333882858, 0.090855689, 0.745590594, 0.817341702, 0.087011920, 0.160546310, 0.415345196, 0.175461572, 3.440819603, 0.208336688, 1.833271636, 2.183783235, 0.446134909, 2.475039374, 1.566538736, 0.886839452, 0.192790814, 2.910071106, 1.163082090, 0.700168337, 0.628095536, 1.683433609, 0.326647849, 1.505027254, 0.709244532, 0.090130483, 1.733876883, 1.791106060, 1.483518345, 0.806656142, 0.002559691, 0.250612339, 0.665047009, 0.192063370, 0.462457812, 1.534055037, 0.181744265, 0.987930554, 0.942945984, 0.943769307, 0.319857968, 0.391021464, 1.697728186, 0.373492095, 1.798812738, 0.496551487, 0.480822546, 0.733421176, 0.737249271, 0.013362448, 1.343100225, 0.180633059, 0.893504639, 1.635429019, 3.840939633, 1.064201159, 1.292336585, 2.076699583, 0.112648713, 3.287097308, 1.984361935, 0.769430699, 1.943481857, 0.553695515, 0.900427014, 0.618240746, 1.583467962, 1.617647828, 2.022053674, 1.482866347, 0.324673680, 3.962655875, 1.152818131, 0.568594399, 0.990417301, 2.133633081, 1.435067333, 1.404984828, 0.438741200, 2.144196713, 0.585965825, 0.239035694, 0.178914244, 2.684646233, 0.514393762, 1.038429367, 0.533911966, 2.749416188, 0.872397132, 0.505439471, 1.475022686, 0.862180884, 0.660766382, 1.385093522

Therefore, we observe the record values from the simulated data as follows: 1.179469, 3.151606, 3.440820, 3.840940, 3.962656.

Here then, for the simulated records, with n = 2, 3, 4, 5 MSE's of the point estimations and Length of the Shortest interval estimation and interval with equal tails are obtained and presented in tables 1, 2 and 3, respectively.

CONCLUSION

In this paper, the MLE, MME and MRE estimations based on upper record range statistic were obtained. Later MRE estimations were obtained for three types of loss functions. On the other hand, the shortest interval confidence based on $R_{U,R}$ was obtained. To get this interval confidence, the researcher encountered equations (16) witch Tate and Klett (1959) they have solved numerically. Using these results, the shortest confidence interval was obtained and shown in table 2. Also, the

Table 1. MSE's of the estimators $\hat{\delta}_{MBURR}$, $\hat{\delta}_{MRE,1}$, $\hat{\delta}_{MRE,3}$.

Number of Record	Estimator	Estimated Value	Bias	MSE
n=2	$\hat{\delta}_{MBURR}$	1.9721369	0	3.8893240
3		1.1306752	0	0.6392132
4		0.8871568	0	0.2623491
5		0.6957966	0	0.1210332
n=2	$\hat{\delta}_{MRE,1}$	0.9860685	-0.4930342	0.48616550
3		0.7537834	-0.2512611	0.18939649
4		0.6653676	-0.1663419	0.11067851
5		0.5566373	-0.1113275	0.06196902
n=2	$\hat{\delta}_{MRE,3}$	2.8451921	1.25955245	18.4354005
3		1.3473676	0.25822124	1.3556359
4		0.9952918	0.12131555	0.4303215
5		0.7579359	0.06768867	0.1749957

Table 2. Shortest interval estimation and Lengths based on upper record range statistic.

Number of Record	Confidence level	Lower limit	Upper limit	Length
n=2	90%	0.2190326	18.431186	18.212153
3		0.2497874	4.282455	4.032668
4		0.2678357	2.447329	2.179493
5		0.2525520	1.624649	1.372097
n=2	95%	0.1836152	38.480720	38.297105
3		0.2147205	6.386191	6.171471
4		0.2340680	3.279086	3.045018
5		0.2234172	2.059560	1.836143
n=2	99%	0.1353737	196.232528	196.097155
3		0.1646996	15.233077	15.068378
4		0.1842307	6.118323	5.934092
5		0.1792672	3.394751	3.215484

Table 3. Interval with equal tail probabilities and Lengths based on upper record range statistic.

Number of Record	Confidence level	Lower limit	Upper limit	Length
n=2	90%	0.6583155	38.448240	37.789925
3		0.4766895	6.363521	5.886831
4		0.4227379	3.254859	2.832121
5		0.3589515	2.036997	1.678046
n=2	95%	0.5346168	77.895247	77.360630
3		0.4058677	9.336349	8.930481
4		0.3683855	4.301908	3.933522
5		0.3174518	2.553698	2.236246
n=2	99%	0.3722195	393.440489	393.068270
3		0.3043487	21.849947	21.545598
4		0.2869884	7.877356	7.590368
5		0.2535361	4.140374	3.886838

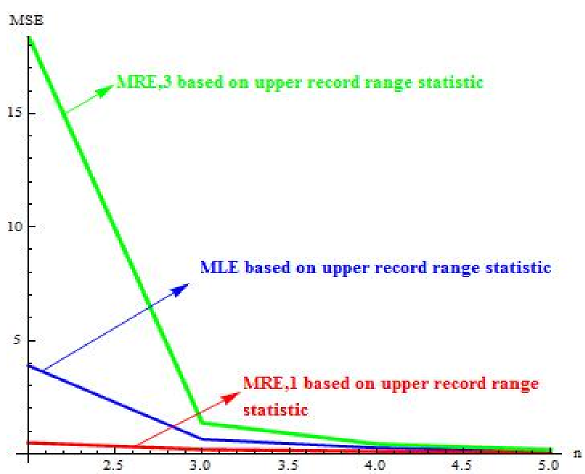

Fig. 1. MSE's of the estimators $\hat{\delta}_{MBURR}$, $\hat{\delta}_{MRE,1}$, $\hat{\delta}_{MRE,3}$ based on upper record range statistic.

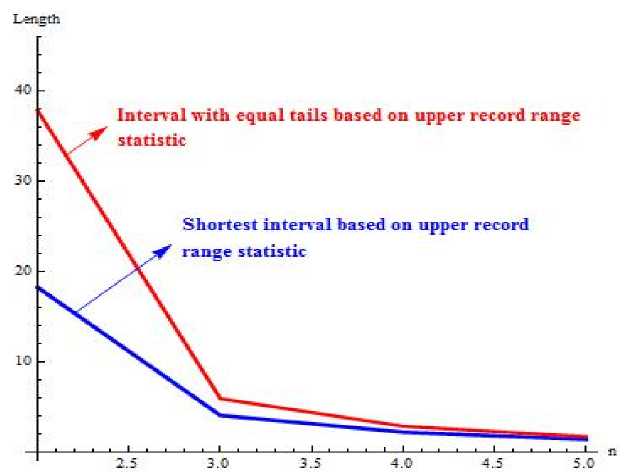

Fig. 2. Length's of the Shortest Interval and interval with equal tails Estimations based on upper record range statistic for 90% confidence.

theoretical results of the study are shown and explained numerically by simulation in the following ways. Table 1 shows that MRE estimation based on $R_{U,R}$ under quadratic loss function has the lowest MSE in comparison with MLE estimation based on $R_{U,R}$ with MRE estimation under squared error loss function. This comparison is shown more vividly in figure 1. In table 2, Shortest interval confidences and their Lengths for records number 2, 3, 4, 5 and confidence ratio 90, 95, 99 levels have been obtained. The longer the "n", the shorter the interval distance. Table 3, shows interval with equal tails and their

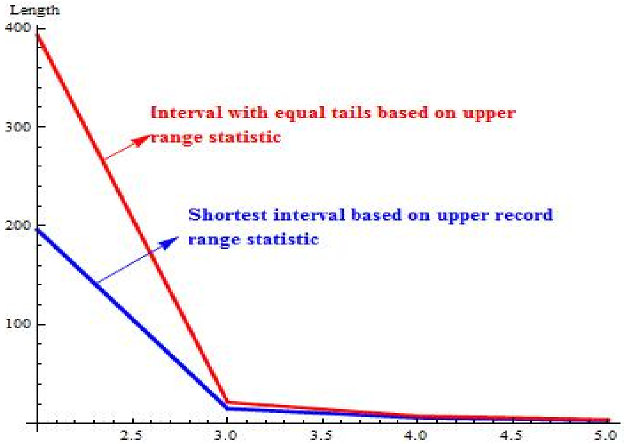

Fig. 3. Length's of the Shortest Interval and interval with equal tails Estimations based on upper record range statistic for 95% confidence.

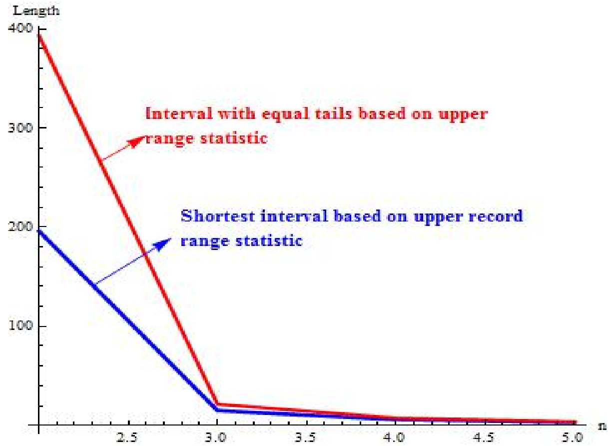

Fig. 4. Length's of the Shortest Interval and interval with equal tails Estimations based on upper record range statistic for 99% confidence.

Lengths for records number 2, 3, 4, 5 and confidence ratio of 90, 95, 99 levels. By comparing tables 2 and 3, the researcher reaches the point that shortest interval based on upper record range have the shorter length than the interval confidence with equal tails. This comparison has been done in figures 2, 3, 4 for different confidence levels.

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