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Short Communication

A NOTE ON LINEAR FUNCTIONAL IN A^p SPACE

Maher MH Marzuq
School of Natural Resources Engineering and Management
Department of Water Engineering and Management
German Jordanian University, PO Box 35247, Amman 111 8, Jordan

ABSTRACT

In this paper we will generalize theorem 9 of Hahn and Mitchell (1969) in bounded symmetric domain on Hardy Space to Bergman Space.

1. Definition and Preliminary Results.

Let D be a bounded symmetric domain in the complex vector space $C^N\left(N>1\right)$ in the cananical Harisch Chandra realization. It is known that D is circular and star-shaped with respect to $0\in D$ and has a Bergman-Silov boundary b, which is circular and measurable. Let Γ be the group of holomorphic automorphisms of D and Γ_0 its isotropy subgroup with respect to 0. The group Γ is transitive on D and the holomorphic automorphisms extend continuous to the topological boundary of D.
The group Γ_0 is transitive on b and b has a unique normalization Γ_0 invariant measure μ which is given by $d\mu_t = V^{-1}ds_t$, V the Euclidean volume of b and ds_t the Euclidean volume element at t (Koranyi and Wolf, 1965).

et H(D) denotes the class of holomorphic functions on as $A^p\left(0<p<\infty\right)$ D, we define the Bergman space follows:

$$A^p = A^p(D) = \left\{ f : f \in H(D) \ and \ \|f\|_{A^p} = \left(\frac{1}{V}\int |f(z)|^p \, dv_z\right)^{\frac{1}{p}} < \infty \right\},$$

or equivalently (Marzuq, 1984a) as,

$$A'^p = A'^p(D) = \left\{ f : f \in H(D) \ and \ \|f\|_{A'^p} = \sup_{0<r<1}\left(\frac{1}{V}\int_D |f(rz)|^p \, dv_z\right)^{\frac{1}{p}} \right\}.$$

In the rest of the paper C is a positive constant not depending on the function, and not necessarily the same at each occurrence.

2. Let T be a linear functional on A^p.

Then, $T\in(A^p)^*$ if and only if it is bounded on the sphere in A^p
Topologies $(A^p)^*$ by setting,

$\|T\| = \sup_{\|f\|_{A^p=1}} |T(f)|$, $(A^p)^*$ is Banach Space (Rudin, 1974).

For, z ∈D set $\gamma_z(f) = f(z)$ (2.1)

Marzuq (1984b) studied linear functional in A^p space.

3. Weak convergence.

Let $\{f_n\}$ be a sequence $A^p(D)$. Then (f_n) is said to converge weakly to $f\in A^p(D)$, written $f_n \to^w f$, if and only if $\gamma(f_n)\to\gamma(f)$ as n→∞, for every $\gamma\in(A^p)^*$. We call f is the weak limit of $\{f_n\}$.

The weak limit of a weakly convergent sequence is unique, for if $\gamma(f_n)\to\gamma(f)$ and $\gamma(f_n)\to\gamma(g)$ for all $\gamma\in(A^p)^*$ then,

$\gamma(f-g)=\gamma(f-f_n)+\gamma(f_n-g)=0$ as $n\to\infty$.

Thus $\gamma(f-g)=0$ for all $\gamma\in(A^p)^*$, and hence f = g, since if f ≠ g, by Corollary (Marzuq, 1984b), there exist $\gamma\in(A^p)^*$, such that $\gamma(f-g)\neq 0$ which contradicts the conclusion $\gamma(f-g)=0$ for all $\gamma\in(A^p)^*$.

We have the following theorem which generalizes theorem 9 (Han and Mitchell, 1969).

Theorem1. Let $f_n \to^w f$, where $f_n\in A^p$ then, $\lim_{n\to\infty} f_n(z) = f(z)$ uniformly or compact bounded symmetric D, where D is an irreducible bounded symmetric domain.

We require the following lemma to prove theorem 1.

Lemma 1: let D be as in theorem1 and X =

Corresponding author email: maher_marzuq@yahoo.com

$\{f \in A^p : \gamma(f)$ is bounded on X for fixed $\gamma \in (A^p)^*\}$. Then there exists B > 0, independent of f, such that,

(i) $|\gamma(f)| \leq B\|\gamma\|$,

(ii) $|f(z)| \leq \dfrac{BC(n_o, p, D)}{(l-r)^{2N_{n_0}/p}}$, $z \in \overline{D_r}$, for $f \in X$.

Proof. The proof of (i) is the same as the proof of theorem 7 (Walters, 1950) (ii) follows from (2.1), and lemma 4 (Marzuq, 1984b).

Proof of theorem 1.

Since $\gamma(f_n) \to \gamma(f)$ for all $\gamma \in (A^p)^*$, then $\gamma(f_n)$ is bounded independently of n. By lemma1 we get,

(ii) $|f_n(z)| \leq \dfrac{BC(n_o, p, D)}{(l-r)^{2N_{n_0}/p}}$ for $z \in \overline{D_r}$, and the bound is independent of n and z.

Since $\gamma_z(f_n) = f_n(z)$ by (2.1), $f_n(z) \to f(z)$ for $z \in \overline{D_r}$.

Hence by Vitali's convergence theorem for C^N [(Valdimirov, 1966), lemma 4] $f_n(z) \to f(z)$ uniformly on compact subset of D.

4. A necessary and sufficient condition for a holomorphic function to belong to the space $A^p (p > 0)$.

We have the following theorem.

Theorem2. Let D be bounded symmetric domain, $z_o \in D_r, o < r < 1,$ and f is holomorphic on D. Then $f \in A^P$ if and only if there exists a constant $C(z_0)$, independent of r, such that,

$$\int_D T\left(z_\circ, \overline{\xi}\right)\left|f_r(\xi)\right|^p dv_\xi \leq C\left(z_\circ\right), \tag{4.1}$$

where,

$$T(z,\overline{\xi}) = \frac{|k(z,\overline{\xi})|^2}{k(z,\overline{z})},$$

and $k(z,\overline{\xi})$ is the Bergman Kernel of D.

We need the following lemma for the proof of theorem 2.

Lemma2. The expression $T(z,\overline{\xi})dv_\xi$ is invariant under Γ, (the group of holomorphic automorphisms of D) .

Proof. Let $\gamma \in \Gamma$ such that $\gamma(z) = 0$ and $\gamma(\xi) = \xi'$. Then,

$$T(z,\overline{\xi})dv_\xi = \frac{|k(o,\xi')|^2 \left|\frac{\delta z'}{\delta z}\right|^2_{z'=o} \left|\frac{\delta\xi'}{\delta\xi}\right|^2 \left|\frac{\delta\xi}{\delta\xi'}\right|^2 dv_{\xi'}}{|k(o,o)|\left|\frac{\delta z'}{\delta z}\right|^2_{z'=o}}$$

$$= \frac{1}{V} dv_{\xi'}, \tag{4.2}$$

(Bergman, 1950).

Proof of theorem2. Assume $f \in A^p$ for fixed $z \in D, \dfrac{k(z,\overline{\zeta})}{k(z,z)}$ is continuous with respect to $\overline{\xi}$ cn $\overline{D}$ (Stoll, 1977).

Therefore,

$$\int_D T\left(z_\circ, \overline{\xi}\right)\left|f_r\xi\right|^p dv_\xi \leq \max_{\zeta \in \overline{D}} T(z_o,\overline{\xi}) V M_p^{1p}(r,f)$$

$$\leq C(z_o)\|f\|_{A^p} = C(z_o).$$

This proves the necessity of (4.1).

Conversely, assume (4.1) is satisfied.

Then,

$$\frac{1}{V}\int_D \left|f(r\xi')\right|^p dv_{\xi_i} = \int_D T(o,\xi')\left|f(r\xi')\right|^p dv_{\xi_i}.$$

For, $\xi \in D$, there exists a holomorphic automorphisms say Y_ξ such that $Y_\xi(z_0) = 0$ which maps ξ into ξ' then by (4.2) we get,

$$\frac{1}{V}\int_D \left|f(r\xi')\right|^p dv_{\xi^1} = \int_D T(z_0,\overline{\xi})\left|f(r\xi)\right|^p dv_\xi \leq C(z_0).$$

.

Hence,

$$\sup_{o \leq r < 1}\left|\frac{1}{V}\int_D f(r\xi')\right|^p dv_{\xi'} \leq C(z_0) < \infty,$$

and $f \in A^p$.

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