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ON SOME COMBINATORIAL RESULTS OF COLLAPSE IN PARTIAL TRANSFORMATION SEMIGROUPS

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ABSTRACT

This paper considers the semigroup of partial transformation P_n and investigated the elements of collapse given by $c(\alpha) = |C(\alpha)| = |\bigcup_{t \in Im\alpha} \{t\alpha^{-1} \geq 2\}|$. Consequently, the formular for the total number of collapsible elements for $|t\alpha^{-1}| = 2$ and $|t\alpha^{-1}| = 3$ in P_n is presented.

Keywords: Collapse, idempotent, nilpotent, partial transformation semigroup.

INTRODUCTION

The theory of semigroup is a natural generalization of group theory which has been popular topic in algebra for many years. The most natural occuring semigroups of this generalization is the transformation semigroup (Ganyushkin and Mazorchuk, 2009; Higgins, 1992; Howie, 1995).

A set S with a binary operation τ is called a semigroup if τ is associative on S. In other words, a semigroup is an algebraic system consisting of a non-empty set together with an associative binary operation that is it is obtained from the set of all transformation on a given set. If γ and λ are transformations on S then, their product $\gamma\lambda$ is defined as;

$$(x)(\gamma\lambda) = ((x)\gamma)\lambda \ .$$

Hence we see clearly, that associative law holds for transformation on the set S.

Let $X_n = \{1,2,......,n\}$, then a (partial) transformation $\alpha : Dom\alpha \subseteq X_n \rightarrow Im\alpha$ is said to be full or total if $Dom\alpha = X_n$ otherwise it is called strictly partial. The set of all partial transformation on n-object forms a semigroup under the usual composition of transformations. In Wagner (1952) stated that the composition of partial transformation is a special case of the multiplication of binary relations. He recognized also that the domain of composition of two partial transformations may be the empty set so he introduces an empty transformation to take account of this. With the addition of this empty transformation, the composition of partial transformations of a set becomes an everywhere defined associative binary operation.

The combinatorial properties of S_n (the partial symmetric semigroup) have been studied over long period and many interesting results have emerged. In particular, the number of permutations of X_n having exactly k fixed points and their generating functions are known.

Various enumerative problems of an essentially combinatorial nature have been considered for certain classes of semigroups of transformations. For example, it is well known that P_n has order $(n+1)^n$. Also the number of idempotents and nilpotent for P_n are given by

$$|E(P_n)| = \sum_{r=0}^{n} \binom{n}{r} (r+1)^{n-r} \text{ and } |N(P_n)| = (n+1)^{n-1}$$

respectively. The first is usually connected to Garba (1990) and the second can easily be deduced from Laradji and Umar (2004a,b).

A study on partial transformation semigroup which we might find interesting in this paper is knowing the total number of collapsible elements $|c(\alpha)|$. In Garba (1990), Howiedefined a collapsible element as $c(\alpha) = \bigcup_{t \in Im\alpha} \{t\alpha^{-1} : |t\alpha^{-1}| \geq 2\}$.

The collapsible element for $|t\alpha^{-1}| = 2$ and $|t\alpha^{-1}| = 3$ from $n \geq 2$ in T_n was studied by Adeniji and Makanjuola (2008). Other interesting results concerning partial transformation semigroup can be found in Howie

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(1966). Again for a non-collapsible elements of P_n that is the partial symmetric semigroup S_n we have $(n+1)!$ for $c(\alpha)$, this is a case where no image appears more than once.
In this paper we investigate collapse in partial transformation semigroup and obtain formular for the total number of collapsible element for $|t\alpha^{-1}|=2$ and $|t\alpha^{-1}|=3$ for $n\geq 2$ in P_n. Finally, we generalize the formular $|t\alpha^{-1}|=2$ and $|t\alpha^{-1}|=3$.

2 PRELIMINARIES

For standard terms and concepts in transformation semigroup theory see Higgins (1992, 1993) and Howie (1995). We now recall notations and definitions that will be used in this paper.

Definition 2.1. Consider $X_n = \{1, 2, \ldots \ldots n\}$ and let $\alpha: X_n \to X_n$ be a partial transformation. By Domα and Imα we mean the domain of the mapping α and image set or range of the mappingα respectively.
As in Umar (1998, 2010) let $\alpha \in P_n$ be denoted by$\alpha = \begin{pmatrix} T_1\, T_2 \ldots \ldots T_r \\ t_1\, t_2 \ldots \ldots t_r \end{pmatrix}$, where $T_1, T_2, \ldots \ldots, T_r$ are pairwise disjoint subsets of X_n called the blocks of α with $T_i\alpha = t_i$. We can be easily seen that the image of α is $Im\alpha = \{t_1, t_2, \ldots \ldots, t_r\}$ and $T_1 \cup T_2 \cup \ldots \ldots \cup T_r = X_n$. If $t_i \in T_i$, we say that T_i is stationary. Otherwise it is non-stationary. If we denote by $F(\alpha)$ the set $\{T_1 \in X_n : T_1\alpha = T_1\}$ and by $f(\alpha)$ its cardinal, then the number of stationary blocks of α is equal to $f(\alpha)$. The element α is idempotent if and only if all the blocks of α are stationary or if $f(\alpha) = |Im\alpha|$.

Proposition 2.2. The set P_n contains $(n+1)^n$ elements.
Proof. Each element $\alpha \in P_n$ is uniquely defined by $\alpha = \begin{pmatrix} 1\ 2 \ldots \ldots n \\ k_1 k_2 \ \ldots \ldots k_n \end{pmatrix}$ where $k_i = \alpha(i)$ if $i \in dom(\alpha)$ and $k_i = \emptyset$ if $i \in dom(\alpha)$ again the element k_i can be independently chosen from the set $N \cup \{\emptyset\}$. Hence the product rule implies $|P_n| = (n+1)^n$ elements.

Lemma 2.3. For all natural numbers n and t we have $\sum_{k=t}^{n} \binom{k-1}{t-1} = \binom{n}{t}$

Lemma 2.4. For all natural numbers k, n and t we have $\sum_{k=0}^{n} \binom{k+t-2}{k-1} = \binom{n+t-1}{n-1} = \binom{n+t-1}{t}$.

Lemma 2.5. Let $\{P_i\} \in Dom\alpha$ and $p_i \in Im\alpha$ such that $|p_i\alpha - 1| \geq 2$. If $p_1 \neq p_2 \neq p_3 \neq \cdots \neq t_i$ then, $\varphi(n,1) = 0$and $\varphi(n,0) = (n+1)!$ where $i = 1,2,3, \ldots \ldots n, n \in \mathbb{N}$.

3 MAIN RESULT

Our result is shown in the triangle of numbers below and we prove a theorem that establishes the total number of collapsible element for $c(\alpha)$, $|t\alpha^{-1}|=2$ and $|t\alpha^{-1}|=3$ for $n\geq 2$ in P_n.

Triangle of numbers $C(n/q)$

C(n/q)	0	1	2	3	4	5	$\sum C(n;q)$
	2	0					2
	6	0	3				9
	24	0	36	4			64
	120	0	360	80	65		625
	720	0	3600	1200	2250	6	7776

Theorem 3.1 *Let* P_n be a partial transformation semigroup and $q = C(\alpha)$ defined by $c(\alpha) = |C(\alpha)| = |\bigcup\{t\alpha^{-1} \geq 2\}|$, then the following is true:
i) if $q = 0, C(\alpha) = (n+1)!$.
ii) if $q = 2, C(\alpha) = \frac{(n+1)!}{2!}\binom{n}{q}$.
iii) if $q = 3, C(\alpha) = \frac{(n+1)!}{3!}\binom{n}{q}$.

Proof. i) When $q = 0$, the element in the semigroup of partial transformation with zero element will be the permutation of all the element which is $n+1$, adding the zero element. Hence $c(\alpha) = (n+1)!$
ii)When $q = 2$, the number of element will be the number of ways of choosing two element from the n elements which is $\binom{n}{2}$ and the arranging of the elements which is $\frac{n+1}{2}$. So the total number will be $\frac{(n+1)!}{2!}\binom{n}{q}$.
iii) When $q = 3$ the number of element will be the number of ways of combining three elememts from the n elements which is $\binom{n}{3}$ and the arranging of the elements which is $\frac{n+1}{3}$ so that the total number is $\frac{(n+1)!}{3!}\binom{n}{q}$
.

Hence we give a general form of collapse in partial transformation semigroup for $n = 0,2,3$ as $c(\alpha) = \frac{(n+1)!}{q!}\binom{n}{q}$.

CONCLUSION

It has been shown that the number of elements that are collapsible in partial transformation semigroup for values of each of $q = 2$ and $q = 3$ can be calculated using $c(\alpha) = \frac{(n+1)!}{q!}\binom{n}{q}$.

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