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ON THE DYNAMICS OF ZADEH EXTENSIONS AND SET-VALUED INDUCED MAPS

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ABSTRACT

In this paper, we consider discrete time dynamical system generated by a continuous mapping f defined in a metric space X along with the induced set-valued mapping $\overline{f}$ and the fuzzified Zadeh extension $\hat{f}$. We study the asymptotic behaviour of these systems using the concept of bi-shadowing and the limit bi-shadowing property.

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INTRODUCTION

The study of asymptotic behaviour of dynamical systems on infinite-time interval with complicated structure usually involve numerical simulation of the system and consequently, the computed (or pseudo-) trajectories are only an approximated trajectories, and therefore, it is important to know whether to each pseudo-trajectory there always exists a true trajectory nearby. This is the idea of shadowing which is a fundamental property in the theory of dynamical systems. Under certain conditions, such as hyperbolicity, it ensures that numerically computed trajectories are close to true ones.

The concept of shadowing was introduced by Anosov (1969) and Bowen (1970). Since then, shadowing is investigated and developed thoroughly, and various types of shadowing properties were proposed and studied by many authors Lee and Sakai (2005) and Sakai (2003). For more details on various aspects of shadowing, we refer to Palmer (2000) and Pilyugin (1999).

On the other hand, the inverse shadowing tries to answer the question whether to any given true trajectory of the system there exists a pseudo-trajectory within a certain threshold. It has some practical importance especially in validating numerical computations of the system. It was introduced by Corless and Pilyugin (1995) and by Kloeden *et al*. (1999) using the concept of δ-method, see also Choi and Lee (2006) and Pilyugin (2002).

A generalisation of shadowing and inverse shadowing, called bi-shadowing, was introduced by Diamond *et al*. (1995), see also Diamond *et al*. (2012). It was considered for set-valued systems with an application to iterated function system by Al-Badarneh (2014) and for infinite dimensional dynamical systems by Al-Nayef (1997). Bi-shadowing consists of a pre-assigned class of pseudo-trajectories, called the class of comparison mappings. A class of comparison mappings that is commonly used in practice is the class of trajectories generated by continuous mappings that are close to the original system. A property that is important for understanding the numerical approximation of trajectories of dynamical systems is called limit shadowing which was introduced by Eirola *et al*. (1997), see also Pilyugin (1999). It was considered by many authors Lee (2001) and Pilyugin (2007).

In this paper, we shall consider a discrete-time dynamical system f defined on a metric space X along with the system generated by the induced set-valued mapping $\overline{f}$ defined on compact subsets of X. We also discuss the relation between the asymptotic behaviour of these systems and the system generated by the fuzzified Zadeh extension $\hat{f}$ defined on fuzzy sets of X using bi-shadowing and a proposed limit bi-shadowing property.

In the next section, we give some definitions and preliminaries needed throughout the paper. Different results on bi-shadowing property that relates the dynamics

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of f, $\overline{f}$ and $\tilde{f}$ will be given in the section: Dynamics via Bi-shadowing. The section: Dynamics via Limit Bi-shadowing will be devoted to similar results using the limit bi-shadowing property.

DEFINITIONS AND PRELIMINARIES

Throughout this paper, X will always denote a compact metric space with metric d and $f: X \to X$ a continuous mapping. Consider the discrete-time dynamical system defined on X generated by the iterations of f, that is, $x_{n+1} = f(x_n)$ where $x_n = f^n(x_0)$ and $f^n = f \circ \cdots \circ f$ (n times) and identify the mapping f with this system. A sequence $\{x_n\}_{n\in\mathbb{Z}} \subset X$ satisfying $x_{n+1} = f(x_n)$, for $n \in \mathbb{Z}$ is called a (true) trajectory of f, whereas a sequence $\{y_n\}_{n\in\mathbb{Z}} \subset X$ satisfying $d(y_{n+1}, f(y_n)) \le \gamma$ for $n \in \mathbb{Z}$ and for $\gamma > 0$ is called a γ-pseudo-trajectory of f.

We say that f has the shadowing property on X if given $\varepsilon > 0$ there exists $\gamma > 0$ such that for any given γ-pseudo-trajectory $\{y_n\}_{n\in\mathbb{Z}}$ of f there exists a true trajectory $\{x_n\}_{n\in\mathbb{Z}}$ of f satisfying $d(y_n, x_n) \le \varepsilon, n \in \mathbb{Z}$.

Let $K(X)$ denotes the class of all nonempty and compact subsets of X. The Hausdorff separation $H^*(A,B)$ of two subsets $A, B \subseteq X$ is defined by

$$H^*(A,B) = \sup_{x\in A} d(x,B)$$

where $d(x,B) = \inf_{y\in B} d(x,y)$ and the Hausdorff distance $H(A,B)$ of $A, B \subseteq X$ by

$$H(A,B) = \max\{H^*(A,B), H^*(B,A)\}.$$

It is well known that the Hausdorff distance H is a metric on $K(X)$ and that the metric space $(K(X), H)$ is complete whenever (X, d) is. Also, it is compact in case (X, d) is compact. The class of all nonempty upper semicontinuous fuzzy sets $u: X \to [0,1]$ such that $L_\alpha u \in K(X)$ will be denoted by $F(X)$, where the α-cuts $L_\alpha u$ of u are defined by

$$L_\alpha u = \{x \in X : u(x) \ge \alpha\}, \quad \alpha \in [0,1],$$

and the support of u by

$$supp(u) = \overline{\{x \in X : u(x) > 0\}}$$

We define the level wise metric D on the space $F(X)$ by

$$D(u,v) = \sup_{\alpha\in[0,1]} H(L_\alpha u, L_\alpha v), \quad \alpha \in [0,1].$$

The space $(F(X),D)$ is complete whenever (X,d) is complete. For a continuous mapping $f: X \to X$ we define the Zadeh extension $\tilde{f}: F(X) \to F(X)$ by

$$\tilde{f}(u)(x) = \begin{cases} \sup_{z\in f^{-1}(x)} u(z) & if \quad f^{-1}(x) \ne \phi \\ 0 & if \ f^{-1}(x) = \phi \end{cases}$$

It was proved by Roman-Flores and Chalco-Cano (2008) that the mapping $\tilde{f}$ is continuous on $F(X)$ if and only if f is continuous on X. Moreover, the continuous mapping $f: X \to X$ induces a continuous set-valued mapping $\overline{f}: K(X) \to K(X)$ defined on $K(X)$ by

$$\overline{f}(A) = f(A), \quad \text{for } A \in K(X).$$

Some properties of α-levels are given in the following theorem of Roman-Flores and Chalco-Cano (2008).

Theorem 2.1. *The family* $\{L_\alpha : \alpha \in [0,1]\}$ *of α-levels of any fuzzy set* $u \in K(X)$ *satisfies the following properties:*

(i) $L_\beta u \subseteq L_\alpha u \subseteq L_0 u$ for all $0 \le \alpha \le \beta$.

(ii) $u=v$ if and only if $L_\alpha u = L_\alpha v$ for all $\alpha \in [0,1]$.

(iii) $L_\alpha(\tilde{f}(u)) = f(L_\alpha u)$, for all $\alpha \in [0,1]$.

(iv) $(\tilde{f})^n = \tilde{f}^n$.

We use the following metric version of the definition of bi-shadowing which was originally introduced by Diamond *et al*. (1995) for single-valued mappings, see also Diamond et al (2012). The class of all continuous mappings $f: X \to X$ will be denoted by $C(X)$.

Definition 2.1. *A continuous mapping* $f: X \to X$ *is called bi–shadowing with respect to C(X) and with positive parameters α and β if for any given γ-pseudo-trajectory* $\{x_n\}_{n\in\mathbb{Z}}$ *of f with* $0 \le \alpha \le \beta$ *and any* $\varphi \in C(X)$ *satisfying*

$$\gamma + \sup_{x\in X} d(\varphi(x), f(x)) \le \beta \tag{2.1}$$

there exists a true trajectory $\{y_n\}_{n\in\mathbb{Z}}$ *of* φ *such that*

$$d(y_n, x_n) \le \alpha(\gamma + \sup_{x\in X} d(\varphi(x), f(x))), \quad n \in Z.$$

(2.2)

DYNAMICS VIA BI-SHADWING

In this section, we consider the dynamical system generated by a continuous mapping $f: X \to X$ with the associated systems $\bar{f}$ and $\hat{f}$ and study the relation between their asymptotic behaviour using the bi-shadowing property.

The class of all Zadeh extensions $\hat{f}: F(X) \to F(X)$ associated to continuous mappings $f \in C(X)$ will be denoted by $\hat{F}(X)$ and the class of all set-valued induced mappings $\bar{f}(X): K(X) \to K(X)$ will be denoted by $\bar{F}(X)$. That is $\hat{F}(X)$={ $\hat{f}$: $\hat{f}$ is the Zadeh extension associated to a map $f \in C(X)$ }, and $\bar{F}(X)$={ $\bar{f}$: $\bar{f}$ is the induced set-valued map associated to a map $f \in C(X)$}.

To measure the proximity of mappings in $F(X)$ we use

$$D_\infty(\hat{f},\hat{g}) = \sup_{u\in F(X)} D(\hat{f}(u),\hat{g}(u)).$$

Definition 3.1 *The set-valued induced mapping* $\bar{f}: F(X) \to F(X)$ *of the continuous mapping f is called bi-shadowing with respect to the class* $\bar{F}(X)$ *and with positive parameters α and β if for any given γ-pseudo-trajectory* $\{A_n\}_{n\in\mathbb{Z}}$ *of* $\bar{f}$ *with* $0 \le \alpha \le \beta$ *and for any comparison mapping* $\Phi \in \bar{F}(X)$ *satisfying*

$$\gamma + \sup_{A\in K(X)} H(\bar{f}(A),\Phi(A)) \le \beta \tag{3.3}$$

there exists a true trajectory $\{B_n\}_{n\in\mathbb{Z}}$ *of Φ such that*

$$H(B_n, A_n) \le \alpha(\gamma + \sup_{A\in K(X)} H(\bar{f}(A),\Phi(A))), \quad n \in Z \tag{3.4}$$

Definition 3.2 *The Zaheh extension* $\hat{f}: F(X) \to F(X)$ *of the continuous mapping f is called bi–shadowing with respect to the class* $\hat{F}(X)$ *and with positive parameters α and β if for any given γ-pseudo-trajectory* $\{w_n\}_{n\in\mathbb{Z}}$ *of* $\hat{f}$ *with* $0 \le \alpha \le \beta$ *and for any comparison mapping Ψ∈* $\hat{F}(X)$ *satisfying*

$$\gamma + \sup_{u\in F(X)} D(\hat{f}(u),\Psi(u)) \le \beta \tag{3.5}$$

there exists a true trajectory $\{u_n\}_{n\in\mathbb{Z}}$ *of Ψ such that*

$$H(u_n, w_n) \le \alpha(\gamma + \sup_{u\in F(X)} D(\hat{f}(u),\Psi(u))), \quad n \in Z$$
(3.6)

Now, we investigate dynamical behaviour of the set-valued induced map $\bar{f}$ and relate this behaviour with that of the Zadeh extension $\hat{f}$.

Theorem 3.1. *Let* $f: X \to X$ *be a continuous mapping and assume that* $\hat{f}: F(X) \to F(X)$ *is bi-shadowing with respect to the class* $\hat{F}(X)$. *Then the mapping* $\bar{f}: K(X) \to K(X)$ *is bi-shadowing with respect to the class* $\bar{F}(X)$.

Proof: Let $\{A_n\}_{n\in\mathbb{Z}}$ be a given γ-pseudo-trajectory of $\bar{f}: K(X) \to K(X)$ with the property that $H(\bar{f}(A_n), A_{n+1}) < \gamma$. Let $\bar{g} \in \bar{F}(X)$ be a comparison mapping associated to some continuous mapping $g \in C(X)$ such that

$$\gamma + \sup_{A\in K(X)} H(\bar{g}(A), \bar{f}(A)) < \beta. \tag{3.7}$$

Thus, we obtain

$$D(\hat{f}(\chi_{A_n}), \chi_{A_{n+1}}) = \sup_{\alpha\in[0,1]} H(L_\alpha \hat{f}(\chi_{A_n}), L_\alpha \chi_{A_{n+1}})$$
$$= H(\bar{f}(A_n), A_{n+1}) < \gamma,$$

where $\chi_{A_n}: X \to \{0,1\}$ is the characteristic function with value 1 if $x \in A_n$ and 0 if $x \notin A_n$. This means that the sequence $\{\chi_{A_n}\}$ is a γ-pseudo-trajectory of $\hat{f}$. If $\hat{g}$ is the Zadeh extension of g and by the relation (3.7) and Part **(iii)** of Theorem 2.1 we obtain

$$\begin{aligned} D_\infty(\hat{f},\hat{g}) &= \sup_{u\in F(X)} D(\hat{f}(u),\hat{g}(u))) \\ &= \sup_{u\in F(X)} \sup_{\alpha\in[0,1]} H(L_\alpha \hat{f}(u), L_\alpha \hat{g}(u)) \\ &= \sup_{u\in F(X)} \sup_{\alpha\in[0,1]} H(\bar{f}(L_\alpha u), \bar{g}(L_\alpha u)) \\ &\le \beta - \alpha. \end{aligned} \tag{3.8}$$

So that the relation (3.5) is satisfied for the mappings $\hat{f}$ and $\hat{g}$, where $\hat{g}$ here plays the role of the comparison mapping for $\hat{f}$.

Now, since $\hat{f}$ is bi-shadowing with respect to the class $\hat{F}(X)$ with constants α and β and in view of the estimate (3.8), it follows that for the γ-pseudo-trajectory $\{\chi_{A_n}\}_{n\in Z}$ of $\hat{f}$ there exists a true trajectory $\{u_n\}_{n\in Z}$ of $\hat{g}$ such that

$$D(u_n, \chi_{A_n}) \le \alpha(\gamma + D_\infty(\hat{f},\hat{g})), \quad n \in Z.$$

Lemma 3.1. *For the true trajectory* $\{u_n\}_{n\in Z}$ *of* $\hat{g}$, *the sequence* $\{L_0 u_n\}_{n\in Z}$ *is a true trajectory of* $\bar{g}$.

Proof: If $z_n = L_0 u_n, n \in Z$ and since $\{u_n\}_{n\in Z}$ is a true trajectory of $\hat{g}$ then $L_0\, \hat{g}(u_n) = L_0 u_{n+1}$. Hence by Part **(iii)**

of Theorem 2.1 we have $g(L_0u_n) = L_0u_{n+1}$ or $\bar{g}(L_0u_n) = L_0u_{n+1}$. This means that $\bar{g}(z_n) = z_{n+1}$, $n \in Z$. The lemma is proved.

To finish the proof of the theorem, we use the relation (3.9) to obtain

$$\begin{aligned} H(L_0 u_{n+1}, A_{n+1}) &= H(\bar{g}(L_0 u_n), A_{n+1}) \\ &\leq \sup_{\alpha\in[0,1]} H(\bar{g}(L_\alpha u_n), A_{n+1}) \\ &= \sup_{\alpha\in[0,1]} H(g(L_\alpha u_n), A_{n+1}) \\ &= \sup_{\alpha\in[0,1]} H(L_\alpha \hat{g}(u_n), L_\alpha(\chi_{A_{n+1}})) \\ &= \sup_{\alpha\in[0,1]} H(L_\alpha u_{n+1}, L_\alpha(\chi_{A_{n+1}})) \\ &= D(u_{n+1}, \chi_{A_{n+1}}) \\ &\leq \alpha(\gamma + D_\infty(\hat{f}, \hat{g})). \end{aligned}$$

(3.10)

So, we have proved that for the γ-pseudo-trajectory $\{A_n\}_{n\in\mathbb{N}}$ of $\bar{f}$ and for any $\bar{g} \in \bar{F}(X)$ there exists a true trajectory $\{L_0 u_n\}_{n\in Z}$ of $\bar{g}$ such that the relation (3.4) is satisfied. This shows that $\bar{f}$ is bi-shadowing with respect to the class $\bar{F}(X)$ and with constants α and β. The theorem is proved.

Theorem 3.2. *Let $f: X \to X$ be a continuous mapping and assume that $\hat{f}: F(X) \to F(X)$ is bi-shadowing with respect to $\hat{F}(X)$. Then the mapping $f: X \to X$ is bi-shadowing with respect to C(X).*

Proof: Let $\{x_n\}_{n\in\mathbb{N}}$ be a γ-pseudo-trajectory of *f* with $d(f(x_n), x_{n+1}) \leq \gamma$ and let $g \in C(X)$ be a continuous mapping satisfying $\gamma + \sup_{x\in X} d(f(y), g(y)) < \beta$. The sequence $\{\chi_{A_n}\}_{n\in Z}$ where A_n is the singleton set consisting of x_n is a γ-pseudo-trajectory of $\hat{f}$ since

$$\begin{aligned} D(\hat{f}(\chi_{A_n}), \chi_{A_{n+1}}) &= \sup_{\alpha\in[0,1]} H(L_\alpha \hat{f}(\chi_{A_n}), L_\alpha \chi_{A_{n+1}}) \\ &= d(f(x_n), x_{n+1}) \leq \gamma. \end{aligned}$$

Moreover,

$$D_\infty(\hat{f}, \hat{g}) = \sup_{u\in F(X)} \sup_{\alpha\in[0,1]} H(f(L_\alpha u), g(L_\alpha u)) \leq \beta - \alpha. \qquad (3.11)$$

But $\hat{f}$ is bi-shadowing with respect to $\hat{F}(X)$ and with constants α and β, so there exists a true trajectory $\{v_n\}_{n\in\mathbb{N}}$ of $\hat{g}$ such that $\hat{g}(v_n) = v_{n+1}, n \in Z$ with the property that

$$D(v_n, \chi_{A_n}) \leq \alpha(\gamma + D_\infty(\hat{f}, \hat{g})),\ n \in Z. \qquad (3.12)$$

By the compactness of the sets $L_0 v_n$, there exists $y_{n+1} \in L_0 v_n$ for $n \in Z$ such that,

$$\begin{aligned} d(y_{n+1}, x_{n+1}) &\leq H(\bar{g}(L_0 v_n), A_{n+1}) \\ &\leq \sup_{\alpha\in[0,1]} H(\bar{g}(L_\alpha v_n), A_{n+1}) \\ &= \sup_{\alpha\in[0,1]} H(g(L_\alpha v_n), A_{n+1}) \\ &= \sup_{\alpha\in[0,1]} H(L_\alpha \hat{g}(v_n), L_\alpha(\chi_{A_{n+1}})) \\ &= \sup_{\alpha\in[0,1]} H(L_\alpha v_{n+1}, L_\alpha(\chi_{A_{n+1}})) \\ &= D(v_{n+1}, \chi_{A_{n+1}}) \\ &\leq \alpha(\gamma + D_\infty(\hat{f}, \hat{g})). \end{aligned}$$

(3.13)

Thus, in view of the relations (3.11) and (3.13) and for the γ-pseudo-trajectory $\{x_n\}_{n\in\mathbb{N}}$ of *f* and for any $g \in C(X)$ there exists a true trajectory $\{y_n\}_{n\in\mathbb{N}}$ (where $y_{n+1} \in L_0 v_n$, $n \in Z$) of *g* such that the relation (2.2) is satisfied. This shows that *f* is bi-shadowing with respect to the class *C*(*X*) and with constants α and β. The theorem is proved.

Theorem 3.3. *Let $f: X \to X$ be a continuous mapping and assume that $\bar{f}: K(X) \to K(X)$ is bi-shadowing with respect to $\bar{F}(X)$. Then $f: X \to X$ is bi-shadowing with respect to C(X).*

Proof: Let $\{x_n\}_{n\in\mathbb{N}}$ be a γ-pseudo-trajectory of *f* satisfying $d(f(x_n), x_{n+1}) \leq \gamma, n \in Z$ and consider the singleton sets $A_n = \{x_n\}, n \in Z$. Note that

$$H(\bar{f}(A_n), A_{n+1}) = H(f(A_n), A_{n+1}) = d(f(x_n), x_{n+1}) < \gamma.$$

That is, the sequence $\{A_n\}_{n\in\mathbb{N}}$ is a γ-pseudo-trajectory of $\bar{f}$. Since $\bar{f}$ is bi-shadowing with respect to $\bar{F}(X)$ with constants α and β and for any comparison mapping $\bar{g} \in \bar{F}(X)$ satisfying

$$\gamma + \sup_{A\in K(X)} H(\bar{g}(A), \bar{f}(A)) < \beta,$$

there exists a true trajectory $\{B_n\}_{n\in\mathbb{N}}$ of $\bar{g}$ such that

$$H(B_n, A_n) \leq \alpha(\gamma + \sup_{A\in K(X)} H(\bar{g}(A), \bar{f}(A))), \quad n \in Z.$$

Using this and the compactness of B_n it follows the existence of $b_n \in B_n$ such that

$$d(b_n, x_n) \leq \alpha(\gamma + \sup_{A\in K(X)} H(\bar{g}(A), \bar{f}(A))), \quad n \in Z. \qquad (3.14)$$

This means, for each γ-pseudo-trajectory $\{x_n\}_{n\in\mathbb{N}}$ of *f* and for each continuous mapping *g* there exists a true trajectory $\{b_n\}_{n\in\mathbb{N}}$ of *g* that satisfy the relation (2.2).

Thus, f is bi-shadowing with respect to the class $C(X)$ and with constants α and β. The theorem is proved.

DYNAMICS VIA LIMIT BI-SHADWING

In this section we introduce the *limit bi-shadowing property* for dynamical systems and study the asymptotic behaviour of f, $\overline{f}$ and $\hat{f}$ using this property.

We say that f has the limit shadowing property on $Y \subseteq X$, see Pilyugin (1999), if for any given γ-pseudo-trajectory $\{y_n\}_{n\in\mathbb{Z}} \subset Y$ of f with the property

$$d(f(y_n), y_{n+1}) \to 0, n \to \infty$$

there exists a true trajectory $\{x_n\}_{n\in\mathbb{Z}}$ of f such that

$$d(x_n, y_n) \to 0, n \to \infty .$$

We say that f has the limit shadowing property in case $Y = X$.

For the metric space X let $L(X)$ denotes any class of (continuous) mappings $\varphi : X \to X$.

Definition 4.1. *A continuous mapping* $f : X \to X$ *is said to have the one-sided limit bi-shadowing property on* $Y \subseteq X$ *with respect to a class L(X) if there exists δ >0 such that, for any given γ-pseudo-trajectory* $\{y_n\}_{n\in\mathbb{Z}} \subset Y$ *of f with* $0 \le \gamma \le \delta$ *and*

$$d(f(y_n), y_{n+1}) \to 0, n \to \infty ,$$

and for any $\varphi \in L(X)$ *satisfying*

$$\sup_{x\in X} d(\varphi(x), f(x)) \le \delta - \gamma, \tag{4.15}$$

there exists a true trajectory $\{w_n\}_{n\in\mathbb{Z}} \subset Y$ *of* φ *such that*

$$d(w_n, y_n) \to 0, n \to \infty \tag{4.16}$$

We say that f has the limit bi-shadowing property if Y=X.

Remark: We mention the following special cases of the preceding definition. There exists δ > 0 such that:

i) If $\varphi = f$ then for any given γ-pseudo-trajectory $\{y_n\}_{n\in\mathbb{Z}}$ of f with $0 \le \gamma \le \delta$ and $d(f(y_n), y_{n+1}) \to 0, n \to \infty$, there exists a true trajectory $\{w_n\}_{n\in\mathbb{Z}}$ of f such that $d(w_n, y_n) \to 0, n \to \infty$. This is the direct limit shadowing property of f, see Eirola et al (1997) Lee (2001) and Pilyugin (2007), for various types of limit shadowing properties.

ii) If γ=0 then for any given true trajectory $\{x_n\}_{n\in\mathbb{Z}}$ of f and for any $\varphi \in L(X)$, (where $L(X) = C(X)$), satisfying (4.15) there exists a true trajectory $\mathbf{y} = \{y_n\}_{n\in\mathbb{Z}}$ of φ such that $d(x_n, y_n) \to 0$ as $n\to\infty$. Here the true trajectory $\mathbf{y}$ of φ is also a β-pseudo-trajectory of f with $\beta < \delta$, this follows by using (4.15) and the estimate:

$$d(f(w_n), w_{n+1}) \le d(f(w_n), g(w_n)) + d(g(w_n), w_{n+1}) < \delta.$$

We shall call the property mentioned in **ii)** above the *inverse limit shadowing* property. Thus, the definition of limit bi-shadowing captures both the direct limit shadowing property and the inverse limit shadowing property.

Theorem 4.1. *Let* $f \in C(X)$ *and assume that* $\hat{f}$ *has the limit bi-shadowing property with respect to* $\hat{F}(X)$. *Then* $\overline{f}$ *also has the limit bi-shadowing property with respect to* $\overline{F}(X)$.

Proof: Consider δ > 0 and let $\{A_n\}_{n\in\mathbb{Z}}$ be a given γ-pseudo-trajectory of $\overline{f}$ where $A_n \in K(X)$, $n \in Z$ with γ < δ and satisfying

$$H(f(A_n), A_{n+1}) \to 0 ,\quad n \to \infty \tag{4.17}$$

Let $\overline{g} \in \overline{F}(X)$ be a comparison mapping associated to some continuous mapping $g \in C(X)$ and satisfying $\sup_{A\in K(X)} H(\overline{g}(A), \overline{f}(A)) \le \delta - \gamma$. Since $\{A_n\}_{n\in\mathbb{Z}}$ is a γ-pseudo-trajectory of $\overline{f}$, we obtain

$$D(\hat{f}(\chi_{A_n}), \chi_{A_{n+1}}) = H(\overline{f}(A_n), A_{n+1}) \le \gamma.$$

Hence, the sequence $\{\chi_{A_n}\}_{n\in Z}$ is a γ-pseudo-trajectory of $\hat{f}$ and moreover

$$D(\hat{f}(\chi_{A_n}), \chi_{A_{n+1}}) \to 0, \quad n \to \infty.$$

It follows from (3.8) that

$$D_\infty(\hat{f}, \hat{g}) = \sup_{u\in F(X)} \sup_{\alpha\in[0,1]} H(\overline{f}(L_\alpha u), \overline{g}(L_\alpha u)) \le \delta - \gamma.$$

Since $\hat{f}$ has the limit bi-shadowing property with respect to the class $\hat{F}(X)$ there exists a true trajectory $\{u_n\}_{n\in\mathbb{Z}}$ of $\hat{g}$ such that $D(u_n, \chi_{A_n}) \to 0$, $n\to\infty$. Finally, by (3.10) we obtain

$$H(L_0 u_n, A_n) = D(u_n, \chi_{A_n}) \to 0 ,\ n \to \infty$$

Hence, for the γ-pseudo-trajectory $\{A_n\}_{n\in\mathbb{Z}}$ of $\overline{f}$ and for any $\overline{g} \in \overline{F}(X)$ there exists a true trajectory $\{L_0 u_{n-1}\}_{n\in\mathbb{Z}}$ of $\overline{g}$ such that

$$H(L_0 u_n, A_n) \to 0 ,\ n \to \infty .$$

This shows that $\overline{f}$ has the limit bi-shadowing property with respect to the class $\overline{F}(\mathrm{X})$. The theorem is proved.

Theorem 4.2. *Let f be a continuous mapping and assume that $\hat{f}$ has the limit bi-shadowing property with respect to $\hat{F}(\mathrm{X})$. Then the mapping f has the limit bi-shadowing property with respect to C(X).*

Proof: Consider δ > 0 and let $\{x_n\}_{n\in\mathbb{Z}}$ be a γ-pseudo-trajectory of f with γ < δ and $d(f(x_n),x_{n+1})\to 0, n\to\infty$ and $g\in C(X)$ a continuous mapping satisfying

$$\sup_{y\in X} \mathrm{d(g(y),f(y))} < \delta-\gamma$$

The sequence $\{\chi_{A_n}\}_{n\in Z}$ where A_n is the singletons $\{x_n\}$, is a γ-pseudo-trajectory of $\hat{f}$. Consequently,

$$\mathrm{D}_{\infty}(\hat{f},\hat{g}) = \sup_{u\in F(X)} \sup_{\alpha\in[0,1]} \mathrm{H(f(L_{\alpha}u), g(L_{\alpha}u))} \le \delta-\gamma.$$

Since $\hat{f}$ has the bi-shadowing property with respect to $\hat{F}(\mathrm{X})$,then there exists a true trajectory $\{v_n\}_{n\in\mathbb{Z}}$ of $\hat{g}$ such that $\hat{g}(\mathrm{v}_n)=\mathrm{v}_{n+1}$, $n\in Z$ with the property that $D(v_n,\chi_{A_n})\to 0, n\to\infty$. By the compactness of the sets $L_0 v_n$, there exists $y_n\in L_0 v_n$, $n\in Z$, such that, (see (3.13))

$$\mathrm{d(y}_n,\mathrm{x}_n) \le \mathrm{D(v}_n,\chi_{A_n}) \to 0, \quad \mathrm{n}\to\infty.$$

Thus, for the γ-pseudo-trajectory $\{x_n\}_{n\in\mathbb{Z}}$ of *f* and for any $g\in C(X)$ there exists a true trajectory $\{y_n\}_{n\in\mathbb{Z}}$ of *g*, where $y_n\in L_0 v_{n-1}$, $n\in Z$ such that $d(x_n,y_n)\to 0$, $n\to\infty$. This shows that *f* has the limit bi-shadowing property with respect to the class *C*(*X*). The theorem is proved.

Theorem 4.3. *Let f be a continuous mapping and assume that $\overline{f}$ has the limit bi-shadowing property with respect to $\overline{F}(\mathrm{X})$, then f has the limit bi-shadowing property with respect to C(X).*

Proof: Let δ > 0 and $\{x_n\}_{n\in\mathbb{Z}}$ be a γ-pseudo-trajectory of *f* with γ < δ and satisfying $d(f(x_n),x_{n+1})\to 0, n\to\infty$. Consider the singleton sets $A_n=\{x_n\}, n\in Z$. Clearly,

$$\mathrm{H}(\overline{f}(A_n),A_{n+1}) = d(\mathrm{f(x}_n),\mathrm{x}_{n+1}),$$

and hence the sequence $\{A_n\}_{n\in\mathbb{Z}}$ is a γ-pseudo-trajectory of $\overline{f}$ satisfying

$$H(\overline{f}(\mathrm{A}_n),A_{n+1}) \to 0,\ \mathrm{n}\to\infty.$$

Since $\overline{f}$ has the limit bi-shadowing property with respect to $\overline{F}(\mathrm{X})$, then for any $\overline{g}\in\overline{F}(\mathrm{X})$ satisfying $\sup_{A\in K(X)} \mathrm{H}(\overline{f}(\mathrm{A}),\overline{g}(\mathrm{A})) \le \delta-\gamma$, there exists a true trajectory $\{B_n\}_{n\in\mathbb{Z}}$ of $\overline{g}$ such that $H(B_n,A_n)\to 0, n\to\infty$. By the compactness of B_n there exists $b_n\in B_n$ such that

$$d(b_n,x_n)\to 0, n\to\infty.$$

So, for each γ-pseudo-trajectory $\{x_n\}_{n\in\mathbb{Z}}$ of *f* and for each continuous mapping *g* there exists a true trajectory $\{b_n\}_{n\in\mathbb{Z}}$ of *g* such that $d(b_n,x_n)\to 0, n\to\infty$. Therefore, *f* has the limit bi-shadowing property with respect to the class *C*(*X*).The theorem is proved.

CONCLUSION

We have used the propertyof bi-shadowing to study the asymptotic behaviour ofa dynamical system generated by a continuous mapping *f*and have made a comparison with the dynamics of the systemsinduced by the set-valued mapping $\overline{f}$ and the fuzzified Zadeh extension $\hat{f}$ .In this direction, we alsoproposed a new definition of limit bi-shadowing to discuss the relation between the dynamics of the above mentioned systems.

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