Canadian Journal of Pure and Applied Sciences
Vol. 9, No. 2, pp. 3449-3453, June 2015
Online ISSN: 1920-3853; Print ISSN: 1715-9997
Available online at www.cjpas.net

Canadian Journal of pure&applied sciences

ON BI-SHADOWING OF SUBCLASSES OF ALMOST CONTRACTIVE TYPE MAPPINGS

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ABSTRACT

A result on bi-shadowing for continuous single-valued contractions satisfying certain conditions has been given by Al-Badarneh (2015). In this paper, we continue the discussion of asymptotic properties of trajectories for mappings belonging to more subclasses of almost contractions. In particular, we prove that Zamfirescu Mappings, Quasi-Contractions, Generalised Contractions, and Hardy-Rogers Contractions are bi-shadowing.

Keywords: Bi-Shadowing, almost contraction, zamfirescu mapping, quasi-contraction, generalised contraction, hardy-rogers contraction.

2010 AMS Mathematics Subject Classification: 37C50, 47H07

INTRODUCTION

The study of asymptotic behaviour of trajectories of discrete-time dynamical systems usually needs techniques and tools that can capture specific properties of the system. These tools include the properties of shadowing and inverse shadowing which are important especially for validating numerical computations of the system (Palmer, 2000; Pilyugin, 2002; Pilyugin, 1999). Generally, shadowing and inverse shadowing ensures the existence of true trajectories near pseudo-trajectories, and vice versa.

In this paper we shall consider a more general property of shadowing and inverse shadowing, called bi-shadowing. It was introduced by Diamond *et al*. (1995) and applied for systems generated by semi-hyperbolic mappings (Diamond *et al*., 2012). Later, it was considered by Kloeden and Ombach (1997) and applied for infinite dimensional dynamical systems by Al-Nayef *et al*. (1997). It was also established by Al-Badarneh (2015) that β-contractions, Kannan mappings, Chatterjea mappings and Reich mappings are all bi-shadowing.

In the next section, we give some definitions and preliminaries needed throughout the paper. The results of bi-shadowing for the subclasses consisting of Zamfirescu Mappings, Quasi-Contractive Mappings, Generalised Contractive Mappings, and Hardy-Rogers Contractive Mappings will be considered in the Main Section Result.

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DEFINITION AND PRELIMINARIES

Throughout the paper, X will always denote a metric space with metric d. We shall consider a dynamical system on X generated by iterations of a continuous mapping $f : X \to X$ and identify f with the corresponding dynamical system.

A sequence $\{x_n\}_{n=0}^{\infty}$ X is called a (true) trajectory of f if $x_{n+1}=f(x_n)$, for $n = 0,1,2,$. While, a sequence $\{y_n\}_{n=0}^{\infty}$ X is called a γ-pseudo-trajectory of f if $d(y_{n+1},f(y_n))\leq\gamma$ for $n = 0, 1, 2,$ and for $\gamma > 0$.

A mapping $f : X \to X$ is called a k-contraction, or a contraction, if there exists a constant $0 < k < 1$ such that

$$d(f(x),f(y)) \leq k\, d(x,y), \quad \text{for all } x,y \ \ X. \tag{2.1}$$

The mapping f is called (λ, L)-contraction or almost contraction (sometimes weak contraction is used), if there exist constants $0\leq \lambda < 1$ and $L \geq 0$ such that

$$d((f(x)), f(y))\leq \lambda\, d(x,y) + L\, d(y,f(x)) \qquad \text{for all } x, y \quad X.$$

Almost contractive mappings are more general than contractions as every β-contractive mapping is almost contraction with $\lambda=k$ and $L=0$.

We say that f has the shadowing property on X if given $\varepsilon>0$ there exists $\gamma>0$ such that for any given γ-pseudotrajectory $\{y_n\}$ there exists a true trajectory $\{x_n\}$ satisfying

$$d(y_n, x_n) \le \varepsilon, \text{ for all } n = 1, 2, \ldots$$

We now give a definition of the concept of bi-shadowing in the context of a general metric space.

Definition 2.1 *A continuous mapping $f:X\to X$ is called bi–shadowing with respect to a comparison class of mappings C(X) consisting of continuous mappings on X and with positive parameters α and β if for any given γ-pseudo-trajectory $\{y_n\}_{n=0}^{\infty}$ of f with $0 \le \gamma \le \beta$ and any $\phi \in C(X)$ satisfying*

$$\gamma + \sup_{x\varepsilon X} d(\phi(x), f(x)) \le \beta \qquad (2.2)$$

there exists a true trajectory $\{x_n\}_{n=0}^{\infty}$ of ϕ such that

$$d(y_n, x_n) \le \alpha(\gamma + \sup_{x\varepsilon X} d(\phi(x), f(x))), \; n=0,1,2,\ldots \qquad (2.3)$$

The following two theorems of bi-shadowing for contractive mappings and almost contractions in a metric space X are given by Al-Badarneh (2015).

Theorem 2.1 *If $f:X\to X$ is a k-contractive mapping on X, then f is bi-shadowing on X with respect to the comparison class C(X) and with positive parameters α and β given by*

$$\alpha = \frac{2}{1-k} \quad \text{and} \quad \beta = 1-k \qquad (2.4)$$

Theorem 2.2 *Let $f:X\to X$ be a continuous almost contraction with constants $0<\lambda<1$ and $L\ge0$ such that*

$$d((f(x)),f(y)) \le \lambda\, d(x,y) + L\, d(y,f(x)), \quad \text{for all } x,y \in X.$$

Assume that $\lambda+L<1$, and that f satisfying the following two conditions:

i) For every γ-pseudo-trajectory $\{z_n\}_{n=0}^{\infty}$ X of f with $\gamma < (1-\lambda-L)/2$, the following series is convergent:

$$S := \sum_{n=0}^{\infty} d(f(z_n), z_n).$$

ii) For every continuous mapping $g:X\to X$ satisfying $\sup_{x\varepsilon X} d(g(x), f(x)) \le (1-\lambda-L)/2$, the following inequality is satisfied:

$$LS < \gamma + \sup_{x\varepsilon X} d(g(x), f(x)). \qquad (2.5)$$

Then f is bi-shadowing on X with respect to the class C(X) and with parameters α and β given by

$$\alpha = \frac{2}{1-\lambda-L} \quad \text{and} \quad \beta = 1-\lambda-L \qquad (2.6)$$

As a consequence of this theorem, it was established by Al-Badarneh (2015) that a continuous Kannan mapping, Chatterjea mapping and Reich mapping that satisfying certain conditions are all bi-shadowing with appropriate constants.

RESULTS AND DISCUSSION

In this section, we consider various subclasses of almost contractions and prove that the systems generated by mappings in these classes are bi-shadowing.

Zamfirescu Mappings

A mapping $f:X\to X$ is called a Zamfirescu mapping if there exist real numbers β,α and c with $0\le\beta<1$, $0\le\alpha<1/2$ and $0\le c<1/2$ such that for all $x,y \in X$, at least one of the following holds:

i) $d(f(x),f(y)) \le \beta\, d(x,y)$,

ii) $d(f(x),f(y)) \le \alpha\, [d(x,f(x))+d(y,f(y))]$, (3)

iii) $d(f(x),f(y)) \le c\, [d(x,f(y))+d(y,f(x))]$.

Note that the conditions i), ii) *and* iii) *above are equivalent to the following condition.*

$$d\big(f(x), f(y)\big) \le h \max\left\{ d(x,y), \frac{d(x,f(x)) + d(y,f(y))}{2}, \frac{d(x,f(y)) + d(y,f(x))}{2} \right\},$$

for all $x,y \in X$, where $0<h<1$, see Ciric (1974).

It was shown in Berinde (2003) that Zamfirescu (1972) mappings are almost contractions, thus, using Theorem 2.2, we have the following result. (4)

Theorem 3.1 *Let a continuous mapping $f:X\to X$ be a Zamfirescu mapping with nonnegative constants β, α and c and assume that the conditions i) and ii) of Theorem 2.2 are satisfied. Then f is bi-shadowing on X with respect to the class C(X) provided that*

$$\delta := \max\left\{\beta, \frac{\alpha}{1-\alpha}, \frac{c}{1-c}\right\} < 1/5.$$

Here, the positive parameters α and β are given by

$$\alpha = \frac{2}{1-5\delta} \quad \text{and} \quad \beta = 1-5\delta \qquad (3.7)$$

Proof: First, we should consider the cases in i), ii) and iii) above and then unify the three constants in one. By Berinde (2006), we have

$$d(f(x),f(y)) \le 2\delta\, d(x,f(x)) + \delta\, d(x,y). \qquad (5)$$

Thus

$$d(f(x),f(y)) \le 2\delta\, d(x,f(x)) + \delta\, d(x,y) \qquad (8)$$

$\leq 2\delta\, d(x,y)+2\delta\, d(y,f(x))+\delta\, d(x,y)$
$= 3\delta\, d(x,y)+2\delta\, d(y,f(x)).$

This shows that f is almost contraction with $\lambda=3\delta$ and $L=2\delta$. Since the conditions i) and ii) of Theorem 2.2 are assumed to be satisfied, Theorem 2.2 implies that f is bi-shadowing with respect to the class C(X) provided that $\lambda+L=5\delta<1$, that is $\delta<1/5$. The values in (3.7) are easily obtained by substituting the values of $\lambda=3\delta$ and $L=2\delta$ in the relations (2.6). This ends the proof of the theorem.

Quasi-Contractions

A mapping $f: X \to X$ is called quasi-contraction, see Ciric (1974), if there exists $0<h<1$ such that

$d(f(x), f(y)) \leq h\, M(x,y)$, for all $x, y \in X$, where

$$M(x,y)=\{d(x,y),d(x,f(x)),d(y,f(y)),d(x,f(y)),d(y,f(x))\}. \quad (3.8)$$

The following theorem of Berinde (2004) implies that quasi contractions are almost contractions for the value of the constant h satisfying $0<h<1/2$.

Theorem 3.2 *Any quasi-contraction with $0<h<1/2$ is an almost contraction.*

Using Theorems 3.2 and 2.2, we have the following result.

Theorem 3.3 *Let a continuous mapping $f:X\to X$ be quasi-contraction with $0<h<1/2$ and assume that the conditions i) and ii) of Theorem 2.2 are satisfied. Then f is bi-shadowing on X with respect to the class C(X).*

Proof: For a continuous quasi-contraction $f:X\to X$, Theorem 3.2 implies that f is almost contraction with appropriate values of λ and L. It follows from the proof of Proposition 3 of Berinde (2004) that the values of λ and L depend on what the maximum value in (3.2) is. For example, if $M(x,y)=d(x,f(y))$, then

$$d(f(x),f(y))\leq \frac{h}{1-h}d(x,y)+\frac{h}{1-h}d(y,f(x)).$$

Thus $\lambda=L=\dfrac{h}{1-h}$ and since the conditions i) and ii) of Theorem 2.2 are assumed to be satisfied, Theorem 2.2 implies that f is bi-shadowing with respect to the class $C(X)$ provided that $\lambda+L=\dfrac{2h}{1-h}<1$, that is $h<1/3$. In this case, the values of $\tilde{\alpha}$ and $\tilde{\beta}$ are:

$$\tilde{\alpha}=2\frac{1-\mathrm{h}}{1-3h} \quad and \quad \tilde{\beta}=\frac{1-3\mathrm{h}}{1-h}.$$

The other cases are treated similarly. The theorem is proved.

Generalised Contractions

A mapping $f:X\to X$ is called generalised contraction, or Ciríc contraction, see (Ciríc, 1971), if there exist nonnegative constants α,β,γ and δ with $\alpha+\beta+\gamma+2\delta<1$ such that

$$d(f(x),f(y))\leq\alpha\, d(x,y)+\beta\, d(x,f(x))+(y,f(y)) +\delta[d(x,f(y))+d(y,f(x))],$$

for all $x,y \in X$.

We now show that a generalised contraction is almost contraction with appropriate constants.

Lemma 3.1 *Let $f:X\to X$ be a generalised contraction (Ciríc) with $\alpha+\beta+\gamma+2\delta<1$. Then f is almost contraction with constants*

$$\lambda=\frac{\alpha+\beta+\delta}{1-\gamma-\delta} \text{ and } L=\frac{\beta+\gamma+2\delta}{1-\gamma-\delta} \qquad (3.9)$$

Proof: The following estimates are direct.

$$\begin{aligned} d(f(x),f(y)) &\leq \alpha\, d(x,y)+\beta\, d(x,f(x))+\gamma\, d(y,f(y))+ \\ &\qquad \delta\, d(x,f(y))+\delta\, d(y,f(x)) \\ &\leq \alpha\, d(x,y)+\beta\,[d(x,y)+d(y,f(x))]+ \\ &\qquad \delta\,[d(x,y)+d(y,f(y))]+\delta\, d(y,f(x)) \\ &\leq (\alpha+\beta+\delta)\, d(x,y)+(\beta+\gamma+2\delta)\, d(y,f(x))+ \\ &\qquad (\gamma+\delta)\, d(f(x),f(y)). \end{aligned}$$

It follows that

$$d(f(x),f(y))\leq\frac{\alpha+\beta+\delta}{1-\gamma-\delta}d(x,y)+\frac{\beta+\gamma+2\delta}{1-\gamma-\delta}d(y,f(x)).$$

If we take

$$\lambda=\frac{\alpha+\beta+\delta}{1-\gamma-\delta} \quad \text{and} \quad L=\frac{\beta+\gamma+2\delta}{1-\gamma-\delta},$$

then clearly, $L \geq 0$ and $\lambda < 1$ since by assumption $\alpha+\beta+\gamma+2\delta<1$. This shows that f is almost contraction. The lemma is proved.

As a consequence of Lemma 3.1 and Theorem 2.2 we have the following result.

Theorem 3.4 *Let a continuous mapping $f:X\to X$ be a generalised (Ciríc) contraction and assume that the conditions i) and ii) of Theorem 2.2 are satisfied. Then f is bi-shadowing on X with respect to the class C(X) provided that $\alpha+2\beta+2\gamma+4\delta<1$ and with positive parameters:*

$$\tilde{\alpha}=2\frac{1-\gamma-\delta}{1-2\gamma-4\delta-\alpha-2\beta}$$

and

$$\tilde{\beta}=\frac{1-2\gamma-4\delta-\alpha-2\beta}{1-\gamma-\delta} \qquad (3.10)$$

Proof: If f is a generalised contraction such that the conditions i) and ii) of Theorem 2.2 are satisfied and since by Lemma 3.1 a generalised contraction is almost contraction, Theorem 2.2 implies that f is bi-shadowing on X with respect to the class C(X) for the values of λ and L satisfying the relation

$$\lambda+L=\frac{\alpha+2\beta+\gamma+3\delta}{1-\gamma-\delta}<1,$$

that is

$$\alpha+2\beta+2\gamma+4\delta<1.$$

The values in (3.10) can be obtained by substituting the values of λ and L given by (3.9) in the relations (2.6). The theorem is proved.

Hardy-Rogers Contractions

We now consider another subclass of almost contractions. A mapping *f:X→X* is called Hardy-Rogers contraction, see Hardy and Rogers (1973), if there exist nonnegative constants a_1, a_2, a_3, a_4, a_5 with $a_1+a_2+a_3+a_4+a_5<1$ such that

$$d(f(x),f(y)) \le a_1 d(x,y) + a_2 d(x,f(x)) + a_3 d(y,f(y))$$
$$+a_4 d(x,f(\mathrm{y})) + a_5 d(\mathrm{y}, f(x)) \text{ for all } x, y \quad X.$$

Clearly if we take, as a special case, $a_2=a_3=a_4=a_5=0$ then the induced mapping is the usual a_1-contraction mapping. While, if we take $a_1=a_4=a_5=0$ then we obtain a contraction condition in the definition of Kannan mappings, see Kannan (1969), and so on. Thus, Hardy-Rogers mappings are more general than most of the contraction type mappings.

Lemma 3.2 *Let f:X→X be a Hardy-Rogers contraction with* $a_1+a_2+a_3+a_4+a_5<1$. *Then f is almost contraction with constants*

$$\lambda = \frac{a_1+a_2+a_5}{1-a_3-a_5} \quad \text{and} \quad L = \frac{a_2+a_3+a_4+a_5}{1-a_3-a_5} \qquad (3.11)$$

Proof: The proof follows the same lines as in the proof of Lemma 3.1 for generalised contractions (Ciric).

Lemma 3.2 and Theorem 2.2 imply the following result.

Theorem 3.5 *Let a continuous mapping f:X→X be Hardy-Rogers contraction and assume that the conditions i) and ii) of Theorem 2.2 are satisfied. Then f is bi-shadowing on X with respect to the class C(X) provided that* $a_1+2a_2+2a_3+a_4+3a_5<1$ *and with positive parameters*

$$\alpha = 2\frac{1-a_3-a_5}{1-2a_3-a_1-2a_2-a_4-3a_5}, \quad \beta = 2/\alpha \qquad (3.12)$$

Proof: Let *f* be a continuous Hardy-Rogers contraction and assume that the conditions i) and ii) of Theorem 2.2 are satisfied. Lemma 3.2 and Theorem 2.2 imply that f is bi-shadowing on *X* with respect to the class *C(X)* provided that

$$\lambda + L = \frac{a_1+2a_2+a_3+a_4+2a_5}{1-a_3-a_5} < 1,$$

that is

$$a_1+2a_2+2a_3+a_4+3a_5<1.$$

The values in (3.12) are obtained by substituting the values of λ and L of (3.11) in the relations (2.6).This ends the proof of the theorem.

CONCLUSION

We utilised a result on bi-shadowing for contractions that satisfy some conditions given by Al-Badarneh (2015). We studied the asymptotic properties for dynamical systems via bi-shadowing in view of the result mentioned above. In particular, we proved that Zamfirescu Mappings, Quasi-Contractions, Generalised Contractions, and Hardy-Rogers Contractions are all bi-shadowing.

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Received: March 28, 2015; Accepted: April 21, 2015