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GROUP THEORY AND HARMONIC OSCILLATORS IN THE PLANE

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ABSTRACT

We show explicitly in this contribution that with a correct identification of the underlying symmetry group to the physical system that represent a finitely many harmonic oscillators in the Euclidian plane, namely the dynamical symmetry group $U(rd)=U_N(1)\times SU(rd)$, it's possible to remove fully the degeneracy that such systems carry for which, furthermore, there is no need to show the importance in physics. In this group notation, r=2 refers to as the dimension of the plane, while d is the number of particles.

Keywords : Harmonic oscillators, Dynamical symmetry group, Spectrum, Degeneracy.

INTRODUCTION

The physical systems that models the harmonic oscillator are of first importance since, despite it's a matter of extrem simplification from the point of view of the complexity of nature, they take already inside themselves the germs of the most spectacular results obtained for models nearby to reality. Except for the macroscopic oscillators like simple or elastic pendulum and all the other similar systems who are also interesting in physics but whose study we are not going to do in this paper, we will concentrate on microscopic physical systems in the nonrelativistic approximation. We stay then in Quantum Electrodynamic (QED) domain which belongs to Quantum Field Theory, at least in its quantum mechanical limits. Already at the classical level we have the electromagnetic theory of Maxwell in the abelian case whose equations lead to solutions that propagate by oscillating in space-time. At the quantum level one associates to the field a particle (and vice versa) which carries a quantum of energy by oscillating also. In this paper we proceed to a dimensional reduction in making abstraction of the space dependence for the degrees of freedom of the system. One talks about theory in 0+1 space-time dimension. We fall hence into the domain of quantum mechanics with finite number of degrees of freedom.

Moreover, beside each group and its representations in mathematics, an important (sometime highly) physical phenomenon is hidden. Explicitly, the degeneracies that are defined basicly in quantum physics as the set of states which share the same energy level (and thus a priori indiscernible), hide very different representatives of a symmetry group. Identify this group amounts to help oneself to get this magnifying glass which allows to unveil the microscopic system.

Consequently, there has been a great interest in the study of the harmonic oscillator at quantum level and particularly in relation with revolutionary tools skillfully borrowed to the theory of groups and representations. In their contribution, firstly aimed at testing and then showing the facilities offered by the physical projector, Govaerts and Klauder (Govaerts and Klauder, 1999) studied a system consisting of d oscillators ($d < \infty$) in the plane. They have shown that when one takes into account the global symmetry SO(d) in addition to the SO(2) local symmetry also called gauge symmetry, it's possible to eliminate the degeneracies from the system. However, as it will be shown in the next lines, and as these authors have already pointed out in their paper, this manner of removing the degeneracies is not fully effective.

In this contribution, taking into account their modern quantization method as well as their results and the open issues raised in their paper, we prove that in taking into account a wider and then more subtle group than SO(d), i.e. the dynamical symmetry group $U(rd)=U_N(1)\times SU(rd)$, all the quantum states of this physical system are identified through specific quantum numbers characterizing the irreducible represention of this group.

The outline is as follows. In the next section we recall the main results obtained in the paper by Govaerts and Klauder (Govaerts and Klauder, 1999) showing that it still remains a persistent degeneracy in the spectrum. Then in section 3 we identify all the states of the system through the dynamical unitary symmetry group $U(rd)=U_N(1)\times SU(rd)$, where r=2 is the dimension of the space in which the gauge group acts i.e. the plane, while d is the number of particles i.e. the dimension of the global

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symmetry group. Some concluding remarks are finally made in section 4.

HARMONIC OSCILLATORS IN THE PLANE: MAIN RESULTS

In this section, beyond the main results whose account is to be found in the paper by Govaerts and Klauder (Govaerts and Klauder, 1999) we stress two concepts: The Dirac quantization algorithm for constrained systems and the physical projector, two tools which need to be tamed.

Consider in the ordinary two dimensional Euclidean plane a system of d harmonic oscillators ($d < \infty$) of identical mass normalized to unity (for simplicity without loss of physical content). The Lagrange function describing such system can be written as follows, with a,b taking their value in the set $\{1, 2\}$,

$$\frac{1}{2g^2}\left(\dot{q}_i^a - \lambda\varepsilon^{ab} q_i^b\right)^2 - V\left(q_i^a\right) \quad i = 1,2,\ldots,d \quad (1)$$

The degrees of freedom of the system are given by the real variable *λ(t)* and the set of 2d real variables q(t) depending only of time and whose dynamics is described by the above Lagrangian. ε^{ab} is the two dimensional antisymmetric tensor such as $\varepsilon^{12}=-\varepsilon^{21}=1$. V(q) is the quadratic function describing the harmonic interaction,

$$V\left(q_i^a\right) = \frac{1}{2}\omega^2 q_i^a q_i^a. \quad (2)$$

The model is then gauge invariant SO(2) and possesses a global symmetry SO(d) associated to the *a priori* indiscerniblity of particles, justifying hence the name given to it: model SO(2)× SO(d) or simply 2 × d. In fact, the above model represents physically a dimensional reduction from (D-1)+1 to 0 + 1 space-time dimensions of some pure gauge theory of SO(2) local symmetry (abelian) with addition of a mass term which is also properly gauge invariant. Indeed, let's consider the Yang-Mills Lagrangian density in some *D*-dimensional Minkowski space-time endowed with the metric structure $\eta_{\mu\nu}$=diag(+,--…-), given by

$$L=-(1/4)\, F_{\mu\nu}{}^a F_a{}^{\mu\nu}, \qquad F_{\mu\nu}{}^a = \partial_\mu A_\nu{}^a - \partial_\nu A_\mu{}^a - g f^{abc} A_\mu{}^b A_\nu{}^c, \quad (3)$$

where $F_{\mu\nu}$ is the electromagnetic tensor deriving from the gauge field $A_\mu{}^a$. a and μ are the Lie algebra index associated to some an *a priori* non-abelian group and the space-time index, respectively. g is the coupling constant and f^{abc} is the structure constant of the considered group. Then, in the limits of the abelian theory[1], the dimensional reduction transforms the variables as follows

$$A_\mu{}^a(\mathbf{x}, t) \rightarrow A_\mu{}^a(t)\ \{A_i{}^a(t) \equiv q_i{}^a(t),\ A_0{}^a(t) \equiv \lambda^a(t)\}. \quad (4)$$

The equations of motion are established from Lagrange-Euler formula $d/dt\{\partial L/\partial(\partial_t q_i)\} - \partial L/\partial q_i = 0$. Specifically, with the gauge condition λ(t)=0, we obtain the following equations which characterize the dynamics of a set of d oscillators constrained to have a vanishing angular momentum in the plane,

$$\partial_t^2 q_i^a = -g^2\omega^2 q_i^a, \quad \varepsilon^{ab} q_i^a \partial_t q_i^b = 0. \quad (5)$$

Note that from the point of view of group theory, this constraint is predictible. Indeed, the SO(2) gauge invariance implies, from the Noether theorem, that the angular momentum is conserved and moreover vanishes for gauge invariant configurations. The classical hamiltonian formulation with the appropriated symplectic structure, using the Dirac algorithm for constrained or singular systems (Govaerts, 1991) presents as follows :

$$H = H_o + \lambda(t)\phi,\ H_o = \tfrac{1}{2}[\, g^2(p_i^a)^2 + \omega^2 (q_i^a)^2\,], \qquad \phi = \varepsilon^{ab} p_i^a q_i^b,\ \ \{q_i^a, p_j^b\} = \delta^{ab}\delta_{ij}. \quad (6)$$

In this notation, H_o represents the fundamental Hamiltonian while λ turns out to be the Lagrange multiplier for the first class constraint ϕ.

From now, the canonical quantization of the model is rather straightforward. The classical phase space variables may be put in correspondence with quantum quantities[2] which must be self-adjoint for an unitary time evolution of the system. However this simplistic way will not be exclusively the only one to be followed, because the aim is to work out the physical spectrum from the wide set of states for the system. For this purpose, the Klauder's physical projector (Klauder, 1997, 1999, 2001) has proved to be particulary usefull. An educational account of the construction and the advantage offered by this projector is given in (Goverts and Klauder, 1999). We won't com back to this detailed development, but we give only the results with emphasize on some unavoidable details.

The first step in a quantization procedure, having in hand the quantum cartesian basis, is to identify an appropriate Hilbert space (quantum space) on which the spectrum could be easily reached. The Fock basis is a natural choice for harmonic systems. Here, this basis is extended to its

1 In this case, the term $gf^{abc}A_\mu{}^a A_\nu{}^c$ vanishes.

2 We will omit the hat over the quantum operators corresponding to classical variables.

helicity sector exploiting the advantage to be in the plane. Moreover, for technical reasons, the coherent state helicity basis is used. This quantum cartesian basis is obtained through the Heisenberg algebra spanned by the following relations,

$$(q_i^a)^\dagger = q_i^a \;,\; (p_i^b)^\dagger = p_i^b \;,\; \{q_i^a \;,\; p_i^b\} = i\hbar\delta_{ab}\delta^{ij} \;. \qquad (7)$$

The quantum composite operators linked to the classical phase space variables are given by

$$H = H_0 + \lambda(t)\phi,\;\; H_0 = \tfrac{1}{2}\, g^2\, p_i^a p_i^a + \tfrac{1}{2}\,\omega^2 q_i^a q_i^a \;, \quad \phi = \varepsilon_{ab}\, p_i^a q_i^b \;. \qquad (8)$$

Remark that these quantities do not suffer of any ambiguities related to the physical concept of normal ordering for operators that not commute. Moreover, the gauge invariance of the system is ensured since $[H_0 , \phi] = 0$.

The annihilation and creation operators in the helicity basis write as follows

$$\alpha_i^{\pm} = -(1/\sqrt{2})[-\alpha_i^1 \pm i\alpha_i^2],\; \alpha_i^{\pm\dagger} = (1/\sqrt{2})[\alpha_i^{1\dagger} \pm i\alpha_i^{2\dagger}] \;,\;\; \alpha_i^a = (\omega/2\hbar g)^{\frac{1}{2}}[q_i^a + i(g/\omega)p_i^a], \qquad (9)$$

with the following commutators

$$[\alpha_i^+ \;,\; \alpha_j^{+\dagger}]= \delta_{ij} = [\alpha_i^- \;,\; \alpha_j^{-\dagger}], \qquad (10)$$

as well as the Hamiltonian and the gauge generator given by

$$H_0 = \hbar g\omega[\, \alpha_i^{+\dagger}\, \alpha_i^+ + \alpha_i^{-\dagger}\, \alpha_i^- + d]= \hbar g\omega[N + d],\;\; \phi = -\hbar[\alpha_i^{+\dagger}\, \alpha_i^+ - \alpha_i^{-\dagger}\, \alpha_i^-]. \qquad (11)$$

The Fock helicity orthonormalized basis is thus spanned by the following kets

$$|n_i^{\pm} > = \Pi^d_{i=1}\, (1/n_i^+!n_i^-!)^{\frac{1}{2}}(\alpha_i^{+\dagger})^{ni+}\, (\alpha_i^{-\dagger})^{ni-}\, |0 >, \qquad (12)$$

showing that the Hamiltonian as well as the unique first class constraint are diagonalized, as follows

$$H_0\, |n_i^{\pm} > = \hbar g\omega[\Sigma^d_{i=1}\, (n_i^+ + n_i^-) + d]|n_i^{\pm} > , \quad \phi|n_i^{\pm}> = -\hbar(n_i^+ - n_i^-)|n_i^{\pm} > . \qquad (13)$$

At this step of the quantisation procedure, one can already clearly see that the physical states of the system i.e. the states annihilated by the first class constraint ϕ, are those for which the sum of the right helicities equals the sum of the left helicities, the so called matching condition,

$$\Sigma^d_{i=1}\{n_i^+\} = n = \Sigma^d_{i=1}\{n_i^-\}, \qquad (14)$$

whereas the energy levels of these states are given by

$$E_n = \hbar g\omega(2n + d),\; n = 0, 1, 2, \ldots . \qquad (15)$$

One can think that the system is hence solved; but two questions readily arise. Are the above states really physical? Otherwise, are there respectfull of the famous matching condition? In the other hand what are the degeneracies for $d \geq 2$?

As we shall see, the answer is negative and the projector evoked above is the appropriated tool for selecting the physical states and highlighting their degeneracies. Furthermore, the coherent states basis allows to take better advantatge of the facilities offered by this operator. The helicity complexe variables to be used for the construction of the helicity coherent states are given by

$$z_i^{\pm} = -(1/\sqrt{2})\, [-z_i^1 \pm iz_i^2], \quad (z_i^{\pm})^{*\dagger} = (1/\sqrt{2})\, [(z_i^1)^{*\dagger} \pm i(z_i^2)^{*\dagger}], \quad z_i^a = (\omega/2\hbar g)^{\frac{1}{2}}[q_i^a + (ig/\omega)p_i^a], \qquad (16)$$

where $(z_i^{\pm})^*$ stands for the conjugated complexe of $(z_i^{\pm})$, while $(z_i^{\pm})^{*\dagger}$ is the adjoint of $(z_i^{\pm})^*$. The corresponding helicity states are given by

$$|z_i^{\pm} > = \exp\{-(1/2)|z|^2\}\, \exp\{z_i^+\alpha_i^{+\dagger}\}\, \exp\{z_i^-\alpha_i^{-\dagger}\}|0 > . \qquad (17)$$

Indeed (Govaerts and Klauder, 1999) the physical projector is an operator which, being applied onto any quantum space quantity, constructs a physical (gauge invariant) one by averaging over the manifold of the gauge symmetry group, all finite gauge transformations generated by the first-class constraint of a system.

In the framework of our model where the gauge group is simply SO(2) for which the manifold is the unit circle parmetrised by the rotation angle $0 < \theta < 2\pi$, the physical projection operator is represented as follows

$$P = \frac{1}{2\pi}\int d\theta \exp\left(\frac{-i}{\hbar}\theta\phi\right) \qquad (18)$$

with the fundamental properties

$$P^2 = P, \quad P^\dagger = P. \qquad (19)$$

The physical time propagator of the system then writes

$$U_{phys}\,(t_2 \,,\, t_1\,) = U\,(t_2 \,,\, t_1\,)P = P\, U(t_2 \,,\, t_1\,)P, \quad U\,(t_2,t_1\,) = \exp\{-(i/\hbar)\textstyle\int_{t1}^{t2} dt\, [H_0 + \lambda(t)\phi]\}. \qquad (20)$$

Let us introduce the complexe parameter

$$x = \exp\{(-i/\hbar)(t_2 - t_1)\hbar g\omega\}. \qquad (21)$$

By integrating over the rotation angle θ and after some computations, one gets

$$U_{phys}\,(t_2,t_1) = x^d x^N P, \qquad (22)$$

where N is the standard excitation levels operator. This expression shows that we are finally led to study the operator

$$x^N P, \quad (23)$$

which encodes the physical spectrum of the model.

Hence denoting these physical states of energy $E_n=\hbar g\omega(2n + d)$ by $|E_n , \mu_n>$, μ_n being the degeneracy index , we set

$$P= \Sigma_{En ,\mu n} \{|E_n , \mu_n >< E_n , \mu_n |\},$$
$$< E_n , \mu_n |E_m , \mu_m >= \delta_{nm} \delta_{\mu n,\mu m} , \quad (24)$$

so that we have the following expression for the physical propagator

$$U_{phys} (t_2, t_1) =$$
$$\Sigma_{En ,\mu n} \{\exp[-(i/\hbar)(t_2 -t_1)E_n] |E_n , \mu_n >< E_n , \mu_n|\}=$$
$$= \exp\{-i(t_2 -t_1)g\omega d\}\times$$
$$\times \exp\{-i(t_2-t_1)2n\, g\omega\}|E_n,\mu_n><E_n,\mu_n|. \quad (25)$$

Consequently, the time dependence of $x^d x^N P$ determines the energy levels, while the matrix elements of this operator give the associated wave function.

• For the energy spectrum and their degeneracies it suffices to work out the trace of the operator (23). Indeed, by comparing equations (22) and (25), we obtain

$$\Sigma_{En ,\mu n} \{x^{(2n+d)} |E_n , \mu_n >< E_n , \mu_n|\}=$$
$$Px^dx^N P = x^dx^NP. \quad (26)$$

This shows that the trace of this operator is nothing but the partition function of the spectrum

$$\mathrm{Tr}x^NP = \Sigma^{\infty}_{n=0} \{d_n x^{2n}\}, \quad (27)$$

where the coefficients[3] d_n, n □ *N*, specify the degeneracies of energy levels $E_n = \hbar g\omega(2n+d)$ of physical states.

• Concerning the physical states, their representations in the configuration space in terms of wave functions are generated by the diagonal matrix elements of the operator (23):

$$< z_i^{\pm} |x^N P|z_i^{\pm} > = \Sigma_{n,\mu n} \{x^{2n} | < z_i^{\pm} |E_n , \mu_n > |^2\}. \quad (28)$$

Remark that this expression lets see already that these wave functions will simply be polynomials, showing the interest of the choice of coherent states basis.

3 One must not confuse d_n with d which represents the number of particles. Moreover, throughout the text, *N* stands for the set of the natural numbers.

Coming back to the spectrum, we have, with $0 < \theta < 2\pi$,

$$Trx^N P=\frac{1}{2\pi}\int d\theta\frac{1}{[1-xe^{i\theta}]^d[1-xe^{-i\theta}]^d}. \quad (29)$$

The degeneracies appear immediately in comparing (27) and (29):

$$d_n = [(d - 1 + n)!]^2 [(d - 1)!n!]^{-2},$$
$$E_n = \hbar g\omega(2n + d), \; n = 0, 1, 2, \ldots . \quad (30)$$

It appears through this result that the degeneracies appear for $d \geq 2$ and grow with the number d of particles. So, the first idea for their elimination is to take into account the indiscerniblity of the d harmonic oscillators sharing the same frequency ω in the Euclidean plane. In other words, a global SO(d) symmetry must be added to the gauge symmetry SO(2).

In terms of quantum helicity degrees of freedom previously defined, the $d(d-1)/2$ generetors of SO(d) are given by

$$L_{ij} = i\hbar[\alpha_i^{a\dagger} \alpha_j^a - \alpha_j^{a\dagger} \alpha_i^a] =$$
$$= i\hbar [\alpha_i^{+\dagger} \alpha_j^+ + \alpha_i^{-\dagger} \alpha_j^- - \alpha_j^{+\dagger} \alpha_i^+ - \alpha_j^{-\dagger} \alpha_i^-], \quad (31)$$

with the following algebra

$$[L_{ij} , L_{kl}] = -i\hbar(\delta_{ik} L_{jl} - \delta_{il} L_{jk} - \delta_{jk} L_{il} + \delta_{jl} L_{ik}). \quad (32)$$

Note that L_{ij} are the equivalent of angular momentum operators in the hyperplane of dimension d. Denoting by (T_{ij}) the tensors which allows the matrix represention in the d-dimensional space of the generetors of the SO(d) global symmetry, we can write L_{ij} as follows

$$L_{ij} = \alpha^{\dagger}\cdot(T_{ij})\cdot\alpha, \quad (T_{ij})_{kl} = i\hbar(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}), \quad (33)$$

with the d×d rotation matrix in SO(d) parametrised by the hyperangle ω_{ij} given by

$$R_{kl} (\omega_{ij}) = (e^{-(i/2)\omega ij\, Tij})_{kl} . \quad (34)$$

Finally, these generetors act onto the helicity coherent states and the creation operators as follows

$$e^{-(i/2)\hbar\omega ij\, Lij}|z_i^{\pm}>= |R_{ij} (\omega_{ij})z_j^{\pm}>,$$
$$e^{-(i/2)\hbar\omega ij\, Lij}\alpha_i^{\pm\dagger}\, e^{(i/2)\hbar\omega ij\, Lij}= \alpha_j^{\pm\dagger} R_{ji} (\omega_{ij}). \quad (35)$$

Having set the required elements, the evaluation of the partition function extended to SO(d) (i.e. the SO(d)-valued partition function) becomes possible. We have

$$\mathrm{Tr}\, x^N \exp\{-(i/2\hbar)\omega_{ij} L_{ij}\}P =$$
$$= \int_0^{2\pi} d\theta/2\pi\{ \det [\delta_{ij} - xe^{i\theta} R_{ij} (\omega_{ij})]\times$$
$$\det[1- xe^{-i\theta} R_{ij} (\omega_{ij})] \}^{-1}. \quad (36)$$

However, instead of the evaluation of this expression for arbitrary ω_{ij} , which is absolutely possible, it's better to only consider a maximal commuting subalgebra among the generetors $L_{ij,}$ namely the Cartan subalgebra (Slansky, 1981). In fact, as it has been explained in the paper (Govaerts and Klauder, 1999), representations of compact semi-simple Lie agebras may be characterized by the Dynkin labels of the Dynkin diagram related to the Cartan subalgebra. We have to distinguish the cases whether d is even or odd. Consequently, in order to proceed with the calculation of the SO(d) valued partition function restricted to the Cartan subalgebra, it proves useful to first consider the simple cases with d=1 and d=2, which will display the structure of the general solution.

i) Case d = 1

Here there is no global symmetry since there is only one particle. Consequently, the corresponding partition function is identical to that obtained in relation (29) with d = 1,

$$\mathrm{Tr}\, x^N P = \int_0^{2\pi} d\theta/2\pi\{ [1-xe^{i\theta}][1-xe^{-i\theta}]\}^{-1} = \Sigma^{\infty}_{n=0}\, x^{2n} = (1-x^2)^{-1}. \quad (37)$$

ii) Case d = 2

The global symmetry in the index i = 1, 2 is that of SO(2). Taking then into account the helicity basis, the only generetor of the abelian SO(2) group defining trivially the Cartan algebra gives

$$L_{12} = i\hbar[\alpha_1^{\pm\dagger}\alpha_2^{\pm} - \alpha_2^{\pm\dagger}\alpha_1^{\pm}] = -\hbar[\alpha_+^{\pm\dagger}\alpha_+^{\pm} - \alpha_-^{\pm\dagger}\alpha_-^{\pm}]. \quad (38)$$

This operator acts onto the coherent states as follows

$$e^{-(i\hbar)\omega 12\, L12}|z_{\pm}^{\pm}>=|e^{\pm i\omega 12}\, z_{\pm}^{\pm}>. \quad (39)$$

Hence the expression (36) reduces to

$$\mathrm{Tr}\, x^N \exp\{-(i/\hbar)\omega_{12} L_{12}\}P = \int_0^{2\pi} d\theta/2\pi\, \{[1-xe^{i(\theta+\omega 12)}][1-xe^{i(\theta-\omega 12)}]\times [1-xe^{-i(\theta-\omega 12)}][1-xe^{-i(\theta+\omega 12)}]\}^{-1} = \Sigma^{\infty}_{n=0} x^{2n}\ \Sigma^{+n}_{p=-n}\{(n+1)-|p|\}e^{2ip\omega 12}. \quad (40)$$

Finally, for the d = 2, all the $d_n = (n + 1)^2$ physical states sharing the same energy level E_n may be listed in the one dimensional representations of the global symmetry SO(2) = U (1) indexed by the whole helicity p so that $-n \le p \le n$ with however a persistent degeneracy given by

$$d(n, p) = n + 1 - |p|, \quad (41)$$

for each of these helicity representations, i.e. for each p. Obviously we have the following verification

$$\Sigma^{+n}_{p=-n}\{d(n, p)\} = (n + 1)^2 = d_n,\ n = 0, 1, \ldots. \quad (42)$$

Clearly, the consideration of the global symmetry in the case d = 2 allows to remove only partially the degeneracy, since for a given p there is still $d(n, p) \neq 1$ states sharing the energy E_n. It means that at quantum level, there is a more wider symmetry than the global symmetry SO(d = 2). This is the global dynamical symmetry associated to the group U (rd) = U (2d).

These first two examples are suggestive enough of the general structure of the organization of the system as far as the global SO(d) symmetry is concerned in addition to the local or gauge symmetry. Thus for the case d = 3, we have again one generetor of Cartan, while for d = 4, one could have two generetors of Cartan. We are now able to generalize according to the parity. However we will not do it since it is a matter of technical hability and it has been properly done in (Govaerts and Klauder, 1999).

THE U (rd) = U_N (1) × SU (rd) DYNAMICAL SYMMETRY

This paragraph stands for our contribution to the complete identificaton of the physical states of the gauge invariant SO(2) model. In the previous paragraph the equation (41) shows that the consideration of the global symmetry SO(d) doesn't allow to remove fully the degeneracies, even though they are partially removed. It appears clairly that quantized, the system admits a symmetry more wider than the global symmetry SO(d). This is the dynamical global unitary symmetry U(rd) = U_N(1) × SU(rd) of wich gauge invariant states we are going to identify in the system. In this notation, r = 2 refers to as the dimension of the space in which the gauge group denoted G acts[4] while d is the dimension of the space on which the global symmetry group acts.

Let us begin with the simplest case d = 2 before a generalization.

The model G = [SO(r = 2), d = 2]

The dynamical symmetry of a spherical harmonic oscillator in the plane is rather SU(2) instead of SO(2). It is well known that the group SU (2) possesses three generators T_1 , T_2 and T_3 in the cartesian basis. Their expressions are obtained by means of arbitrary unitary linear combinations of quantum cartesian or helicity coordinates.

It is common (usefull) to redefine the two first generators

4 This space, which is here the ordinary two dimensional Euclidean plane, has nothing to do with the manifold of the associated gauge group which is the unit circle, and thus of dimension 1.

to obtain the helicity generators $T_{\pm}$ associated to the remaining, T_3 ,

$$[T_a , T_b] = i\varepsilon_{abc}T_c, \quad T_{\pm} = T_1 \pm iT_2 ,$$
$$T_1 = (1/2)\,[T_+ + T_-], \quad T_2 = (1/(2i))\,[T_+ - T_-] ,$$
$$[T_+ , T_-] = 2T_3 , \quad [T_3 , T_{\pm}] = \pm T_{\pm} ,$$
$$\mathbf{T}^2 = (1/2)\,(T_+ T_- + T_- T_+) + T_3^2 . \qquad (43)$$

It is also well known that in the representation of spin t, the eigenvalue of the Casimir operator $\mathbf{T}^2 = T_1^2 + T_2^2 + T_3^2$ is t(t+1), while the generator associated to $U_N(1)$ is the excitation levels operator N of quantum numbers n.

Hence, if m is the eigenvalue associated to T_3, one easily shows that $-t \le m \le t$, so that the weight diagram of SU (2) which is a group of rank one is the sector of the m-axis confined between −t and +t. One shows then that the states determined by the kets $T_{\pm}|t, m>$ are eigenstates of T_3 with the eigenvalues (m±1). In abstract, the spin representation is given by

$$\mathbf{T}^2 : \; t(t+1), \; t \,\square\, N, N + 1/2 ,$$
$$T_3 \,|m> = m|m> , \; <m|m'> = \delta_{mm'} ,$$
$$m = -t, -t+1, \cdots , t-1, t \quad , \qquad (44)$$

$$T_{\pm} \,|m> = [t(t + 1) - m(m \pm 1)]^{1/2}|m \pm 1> ,$$
$$T^2 \,|m> = t(t + 1)|m> .$$

This clearly means that starting from the highest weight state m = t and by application of T_- , one falls imediately onto the previous state in the weight diagram and so on. The same considerations is absolutely possible starting from the states of lowest weight by successive applications of the operator T_+ . These facts are fundamental, since it is henceforth possible to identify all the representatives of this symmetry.

For an harmonic oscillator corresponding to the case d = 1, we know that at the excitation level n, the quantum numbers t and m caracterizing SU(2) are given in the helicity basis by

$$|n_+, n_-> = [n_+!n_-!]^{-1/2}(\alpha_+^{\dagger})^{n+}(\alpha_-^{\dagger})^{n-}\,|0 >, \; n = n_+ + n_- , \; m =(1/2)(n_+ - n_-), \quad t =(1/2)\,(n_+ + n_-), \qquad (45)$$

giving hence the indispensable relation between the quantum numbers n, t and m for this case.

In conclusion, for an harmonic oscillator in the ordinary Euclidiean plane and for fixed value of the quatum number $n = n_+ + n_-$, the states $|n_+ , n_- >$ sharing the same energy level $\hbar\omega(n+1)$ span the totaly symmetric representations of SU(2) of dimension n + 1 = 2t + 1. These states of spin t = n/2 are distinguished by their eigenvalues of T_3 , i.e. $(n_+ - n_-)$, and are related by the operators T_+ and T_- . This procedure is resumed until exhaustion of t. In particular, the physical states in the contexte of the considered gauge invariant model here will be those corresponding then to the eigenvalue m = 0.

Let us apply the same analysis to the case d=2 of our model. In this case, the following choices may be done to facilitate the identification of the physical states. *The inclusions of the gauge group SO(2) and that of the global symmetry group SO(d = 2) into SU(r = 2) and SU(d = 2) respectively are chosen such that*

- *ϕ coincides with the generator T_3 of the Cartan subagebra of SU(r = 2).*
- *L_{12} coincides with the generator T_3 of the Cartan subalgebra of SU(d = 2).*

Consequently, the physical states are such that the eigenvalues of T_3 for SU(r = 2) vanish and that of T_3 for SU (d = 2) corresponds to the helicity quantum number p of SO(d = 2). Finally, in addition to the excitation quantum number n, the physical states are charcterized by the quantum numbers of SU(d=2) i.e. the value of the spin t and that of T_3 in SU(d = 2) which is represented here by m.

Let keep ourselves to the concrete determination of the gauge invariant states within the representation of the global dynamical symmetry SU(2) = SU(d = 2). Let us note these states as follows

$$|n, t, p = m > . \qquad (46)$$

The generators of this symmetry SU(2) are given in the appropriated basis by

$$T_1 = (1/2)[\alpha_+^{+\dagger}\,\alpha_-^{+} + \alpha_-^{+\dagger}\,\alpha_+^{+}] + \; + (1/2)\,[\alpha_+^{-\dagger}\,\alpha_-^{-} + \alpha_-^{-\dagger}\,\alpha_+^{-}] ,$$

$$T_2 = -(1/2i)[\alpha_+^{+\dagger}\alpha_-^{+} - \alpha_-^{+\dagger}\alpha_+^{+}] \; -(1/2i)[\alpha_+^{-\dagger}\alpha_-^{-} - \alpha_-^{-\dagger}\alpha_+^{-}] , \qquad (47)$$

$$T_3 = (1/2)[\alpha_+^{+\dagger}\,\alpha_+^{+} - \alpha_-^{+\dagger}\,\alpha_-^{+}] + \; + (1/2)\,[\alpha_+^{-\dagger}\,\alpha_+^{-} - \alpha_-^{-\dagger}\,\alpha_-^{-}],$$

$$T_{\pm} = T_1 \pm iT_2 = \alpha_{\pm}^{+\dagger}\,\alpha_+^{+} + \alpha_{\pm}^{-\dagger}\,\alpha_-^{-} ,$$

while the excitation levels operator also called number operator is given by

$$N = \alpha_+^{+\dagger}\,\alpha_+^{+} + \alpha_-^{+\dagger}\,\alpha_-^{+} + \alpha_+^{-\dagger}\,\alpha_+^{-} + \alpha_-^{-\dagger}\,\alpha_-^{-} . \qquad (48)$$

To tell the truth, one can start from the general Fock states given in Eq. (12) and determine each of the quantum numbers $n_{\pm}^{\pm}$ such that the above two conditions of immersion of SO(2) into SU(2) are realized, begining from the highest weight state for which t = p = n and going down step by step by application of T_- with the intention of identifying all the 2n + 1 physical states associated to this highest weight state, with the quantum number given by

$$n = n_+^+ + n_+^- + n_-^+ + n_-^- . \quad (49)$$

The operation is repeated for the following highest weight states until the display of the $(n+1)^2$ physical states expected through the degeneracy.

However, instead of going down that long path, one can start from a state whose general structure takes already into account these requirements and construct directly the expected particular states.

Hence, the physical states may be represented as follows

$$[(n_{++}+ n_{+-})!(n_{-+} + n_{--})!(n_{++} + n_{-+})!(n_{+-} + n_{--})!]^{-1/2}\times (\alpha_+^{+\dagger}\alpha_+^{-\dagger})^{n_{++}}(\alpha_+^{+\dagger}\alpha_-^{-\dagger})^{n_{+-}}(\alpha_-^{+\dagger}\alpha_+^{-\dagger})^{n_{-+}}(\alpha_-^{+\dagger}\alpha_-^{-\dagger})^{n_{--}}|0 >, \quad (50)$$

so that the quantum number associated to the operator N is given by

$$N = (n_{++} + n_{+-}) + (n_{-+} + n_{--}) + (n_{++} + n_{-+}) + (n_{+-} + n_{--}) = 2n, \quad (51)$$

and that associated to T_3 writes

$$m=p= (1/2) [(n_{++} + n_{+-}) - (n_{-+} + n_{--}) + (n_{++} + n_{-+}) - (n_{+-} + n_{--})] = (n_{++} - n_{--}). \quad (52)$$

From Eq. (42), we know that for a given n, there is $(n + 1)^2$ states distinguished by their quantum numbers t and p (associated to the dynamical symmetry SU (2)) sharing the same energy level whose first highest weight state is given by t = p = n.

By using the following usefull formula, we can identify explicitly the physical states.

$$T_- |t, p >= [(t - p + 1)(t + p)]^{1/2} |t, p - 1 > . \quad (53)$$

1. **The fundamental level n = 0 = N**

The highest weight state which stands at the same time of the singlet of the representation in this case is given by

$$|0, 0, 0 >, \quad (54)$$

such that

$$T_3 |0, 0, 0 >= 0, \; T_+ |0, 0, 0 >= 0 = T_- |0, 0, 0 > . \quad (55)$$

2. **The level n=1, N=2n=2**

i) Maximal weight state t = p = n = 1

$$|1, 1, 1 >= \alpha_+^{+\dagger} \alpha_+^{-\dagger} |0 > . \quad (56)$$

This state is normalized such that T+ |1, 1, 1 >=0.

i)-1 Previous state

The state before |1, 1, 1 > is |1, 1, 0 > such that $T_- |1, 1, 1 >= 2^{1/2}|1, 1, 0 >$. We have

$$|1, 1, 0 >= 2^{-1/2} [\alpha_-^{+\dagger} \alpha_+^{-\dagger} + \alpha_-^{-\dagger} \alpha_+^{+\dagger}] |0, 0, 0 > . \quad (57)$$

i)-2 State before |1, 1, 0 >

The previous state is |1, 1, −1 > such that $T_- |1, 1, 0 >= 2^{½}|1, 1, -1 >$. Consequently, we have

$$|1, 1, -1 >= \alpha_-^{+\dagger} \alpha_-^{-\dagger} |0, 0, 0 > . \quad (58)$$

This state is the last of the subgroup of states characterized by the spin t = 1, since we have $T_- |1, 1, -1 >= 0$.

ii) Following state of highest weight : t = p = 0

It is given by

$$|1, 0, 0 >= 2^{-1/2} [\alpha_-^{+\dagger} \alpha_+^{-\dagger} - \alpha_-^{-\dagger} \alpha_+^{+\dagger}]|0, 0, 0 > . \quad (59)$$

In conclusion at the level n = 1 the set of $4 = (1 + 1)^2$ states sharing the energy level E_n presents as follows

$$\{|1, 1, 1 > |1, 1, 0 >, |1, 1, -1 >, |1, 0, 0 >\} . \quad (60)$$

3. **The level n=2, N=2n= 4**

The construction of the corresponding states follows strictly the same principle as above.

i) State of highest weight t = p = 2

$$|2, 2, 2 >= (1/2) (\alpha_+^{+\dagger} \alpha_+^{-\dagger})^2 |0, 0, 0 > . \quad (61)$$

i)-1 Previous state to |2, 2, 2 >:

$|2, 2, 1 >= (1/2) T_- |2, 2, 2 >$

$$|2, 2, 1 >= (1/2) (\alpha_-^{+\dagger} \alpha_+^{-\dagger} + \alpha_+^{+\dagger} \alpha_-^{-\dagger})\alpha_+^{+\dagger} \alpha_+^{-\dagger} |0, 0, 0 >. \quad (62)$$

i)-2 State before |2, 2, 1 >:

$|2, 2, 0 >= 6^{-1/2}T_- |2, 2, 1>$

$$|2, 2, 0 > = (1/2) 6^{-1/2} \{(\alpha_-^{+\dagger})^2 (\alpha_+^{-\dagger})^2 + (\alpha_+^{+\dagger})^2 (\alpha_-^{-\dagger})^2 + 4\alpha_+^{+\dagger} \alpha_+^{-\dagger} \alpha_-^{+\dagger} \alpha_-^{-\dagger}\}|0, 0, 0 > . \quad (63)$$

i)-3 Previous state to |2, 2, 0 >

$$|2, 2, -1 >= 6^{-1/2} T_- |2, 2, 0 >= (1/2)\{\alpha_-^{+\dagger} \alpha_+^{+\dagger} (\alpha_-^{-\dagger})^2 + \alpha_+^{-\dagger} \alpha_-^{-\dagger} (\alpha_-^{+\dagger})^2 \}|0, 0, 0 > \quad (64)$$

i)-4 Previous state:
$|2, 2, -2 >= (1/2)\, T_- \,|2, 2, -1 >$
$|2, 2, -2 >= (1/2)(\alpha_-^{+\dagger}\, \alpha_-^{-\dagger})^2 \,|0, 0, 0 > .$ (65)

ii) Following state of highest weight: t = p = 1

$|2, 1, 1 >=$
$(1/2)\, (\alpha_-^{+\dagger}\, \alpha_+^{-\dagger} - \alpha_+^{+\dagger}\, \alpha_-^{-\dagger})\alpha_+^{+\dagger}\, \alpha_+^{-\dagger} \,|0, 0, 0 > .$ (66)

ii)-1 Previous state to $|2, 1, 1 >$:
$|2, 1, 0 >= 2^{-1/2}\, T_- \,|2, 1, 1>$
$|2, 1, 0 >=$
$2^{-3/2}\, \{(\alpha_-^{+\dagger})^2\, (\alpha_+^{-\dagger})^2 - (\alpha_+^{+\dagger})^2\, (\alpha_-^{-\dagger})^2 \}|0, 0, 0 > .$ (67)

ii)-2 Previous state to $|2, 1, 0 >$:
$|2, 1, -1 >= 2^{-1/2}\, T_- \,|2, 1, 0 >$

$|2, 1, -1 >=$
$(1/2)\{(\alpha_-^{+\dagger})^2\, \alpha_+^{-\dagger}\, \alpha_-^{-\dagger} - (\alpha_-^{-\dagger})^2\, \alpha_-^{+\dagger}\, \alpha_+^{+\dagger}\}|0, 0, 0 > .$ (68)

iii) Following state of highest weight: t = p = 0

$|2, 0, 0 >= (1/2\sqrt{3})[\, (\alpha_-^{+\dagger})^2\, (\alpha_+^{-\dagger})^2 +$
$+(\alpha_+^{+\dagger})^2\, (\alpha_-^{-\dagger})^2 - 2\alpha_+^{+\dagger}\, \alpha_-^{+\dagger}\, \alpha_+^{-\dagger}\, \alpha_-^{-\dagger}\,]|0, 0, 0 > .$ (69)

One can easily check that $T_- \,|2, 0, 0 >= 0$, showing that there is no state beyond. In conclusion, the set of the nine physical states at the excitation level n = 2 is given by

$$\{|2, 2, 2 >, |2, 2, 1 >, |2, 2, 0 >, |2, 2, -1 >, |2, 2, -2 >, |2, 1, 1 >, |2, 1, 0 >, |2, 1, -1 >, |2, 0, 0 >\} . \quad (70)$$

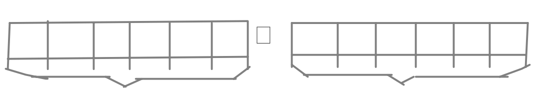
Having hence the general structure of the repartition of the physical states for these two first energy levels, one can generalize for an arbitrary quantum number n without risk of mistake.

4. The level 2n = N

It is now clear that the physical states are given in general, up to normalization factors, in
SU (2) by

$$\prod_{i=1}^{n}(\alpha_{\eta_i}^{+\dagger}\alpha_{\delta_i}^{-\dagger})|0 >=(\prod_{i=1}^{n} \alpha_{\eta_i}^{+\dagger})(\prod_{i=1}^{n} \alpha_{\delta_i}^{-\dagger})\, |0 >, \quad \eta_i\, , \delta_i = \pm \;, \quad (71)$$

where each of the terms ($\prod_{i=1}^{n} \alpha_{\eta_i}^{+\dagger}$):= (I) and ($\prod_{i=1}^{n} \alpha_{\delta_i}^{-\dagger}$):=(II) corresponds to the totaly symmetric representation of n box in terms of the diagram of Young. Thus these (I) × (II) states fall into the following representations


(72)

n n

where, in this tensor product, each representation is characterized by the spin

t = (1/2)n so that one has finally a partition of the set of states whose each value of partition element, for fixed n, gives the state of highest weight

(n/2) □ (n/2) =
n □ (n − 1) □ (n − 2) □ · · · □ 0 . (73)

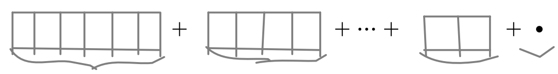
Hence the first highest weight state is characterized by[5] n, the following by n − 1 and so on, and the last by 0.

Furthermore, for a given n, the dimension of the representation is given by

$$(2n + 1) + (2n - 1) + (2n - 3) + \cdots + 1 = (n + 1)^2 , \quad (74)$$

which coincides, as one should expect, with the degeneracy of the physical states at the excitation level n.

In conclusion, at quantum level N marked by N = 2n, all the physical states fall into the totaly symmetric representations symbolized by the diagrams of Young

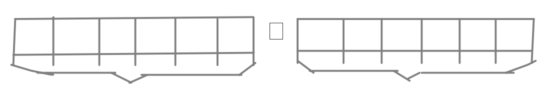

, (75)

2n	2n-2	2n=2	n=0
t=n	t=n-1	t=1	t=0

that one can distinguish by their quantum number p associated to the projection T_3 in these representations.

The general model G = [SO(r = 2), d]
Let us close this paragraph with the general case where there is a finitely many, specifically d particles always within the framework of the gauge group SO(2).

By generalizing the analysis done for the case d = 2, it is obvious that at quantum level N = 2n, all of the physical states fall into the representations of SU(d) given by


, (76)

n n

and since the dimension of each of the terms in the tensor product is given by

$$(1/n!)[d(d + 1) \ldots (d + n - 1)] = [\, (d - 1 + n)!\,] \times [(d - 1)!n!]^{-1} , \quad (77)$$

5 This state, obtained for t = n is accompanied by a set of states obtained by successive applications of T_-, characterized by the same quantum number t but different by their quantum number p.

then the dimension of the representation given by (76) is simply the degeneracy given by

$$d_n = [(d-1+n)!]^2 \times [(d-1)!n!\ ]^{-2}. \qquad (78)$$

It will simply be a matter of reduction of the representation given by (76) and to identify at the end the different states by their specific quantum numbers in SU(d) for each set of the obtained partition.

CONCLUSION

In this paper, we have shown that the global unitary dynamical symmetry is the right one for a skillful description of a finitely many oscillators in the ordinary Euclidean plane. If it's known through various studies devoted to relations between group theory and physics of particles that harmonic oscillators symmetries are the unitary ones, our study based on one of the simplest laboratory model gives an explicit example and shows how does it work. It is possible to apply the same formalism for the case of the SO(3) gauge group for particles moving no more in the plane but in the three dimensional space with spherical symmetry, with -- this is to be noted -- the help of SU(2) coherent states (Avossevou, 2013).

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