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Short Communication

SOME SPECIAL LAPLACE INVERSIONS

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ABSTRACT

In this paper, we establish a method for determining some special Laplace transform inversions, by forming the generating differential equations and solving by other methods. Examples of Cartesian plane methods, axially symmetric methods together with spherically symmetric cases are treated. Correspondence is made between these solutions and the solutions obtained by Laplace transform methods.

Keywords: Laplace transforms, Differential equations, Similarity solutions, Fourier transforms, Bessel functions.

INTRODUCTION

Many difficult Laplace transforms inversions can be obtained when solving initial boundary value problems. Final solutions are the results of inverting the resulting Laplace transforms. Generally speaking these inversions cannot be obtained by standard inversion methods; most powerful of which, the complex Laplace inversion method as the residues of $e^{st} f(s)$ at the poles of $f(s)$. In cases of impossibility, we search for the generating differential equation and attempt solutions by other methods, and if solutions can be obtained correspondence is to be made.

Reference is made to standard textbooks on the Laplace transform and its inversions (Gupta, 1978). Since we restrict ourselves in this paper to use similarity solutions as alternative methods we refer to (Evans, 2010).

APPLICATIONS

We start with the diffusion equation $\frac{\partial c}{\partial t} = D\,\nabla^2 c$, which reduce in one dimensional Cartesian case to:

$$\frac{\partial c}{\partial t} = D\frac{\partial^2 c}{\partial x^2} \tag{1}$$

Together with the initial condition
$c(x,0) = 0, \quad 0 \le x < \infty$

And the boundary conditions
$c(0,t) = c_0$ and $c(\infty,t) = \frac{\partial}{\partial x}\, c(\infty,t) = 0$

This represents diffusion in semi-infinite one dimensional slab; initially at zero concentration c (or temperature). The value of concentration or temperature at the end is a prescribed value c_0. In terms of the similarity variable (Harold, 1979) $\eta = \frac{x}{2\sqrt{Dt}}$ the solution is well known to be (DiBenedetto, 2010).

$$c(x,t) = c_0\ erfc(\eta) \tag{2}$$

Now, let us form the solution of (1) by Laplace transform as

$$s\hat{c} = D\,\frac{d^2\hat{c}}{dx^2}$$

i.e. $\frac{d^2\hat{c}}{dx^2} - \frac{s}{D}\hat{c} = 0$

Which the solution

$$\hat{c}(x,s) = k\,e^{-\sqrt{\frac{s}{D}}\,x}$$

$$\hat{c}(0,s) = k = \frac{c_0}{s}$$

i.e. $$\hat{c}(x,s) = \frac{c_0}{s}\,e^{-\sqrt{\frac{s}{D}}\,x} \tag{3}$$

Obviously, comparing equations (2), (3) we can conclude

$$L^{-1}\left(\frac{1}{s}\,e^{-\sqrt{\frac{s}{D}}\,x}\right) = erfc\left(\frac{x}{2\sqrt{Dt}}\right) \tag{4}$$

$erfc$ is the complementary error function defined by $\frac{2}{\sqrt{\pi}}\int_{\eta}^{\infty} e^{-\eta^2}\,d\eta$

Now we proceed to obtain the solution of the diffusion equation

$$\frac{\partial c}{\partial t} = D\left(\frac{\partial^2 c}{\partial r^2} + \frac{1}{r}\frac{\partial c}{\partial r}\right) \tag{5}$$

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In the interior of the unit cylinder and in its exterior subject to the conditions $c(1,t) = c_0$ and $c(r,0) = 0$ for $t \geq 0$ and all radii r. Using the similarity variable $\eta = \frac{r}{2\sqrt{Dt}}$ the solution of (5) can be obtained as

$$c\left(\eta = \frac{r}{2\sqrt{Dt}}\right) = \alpha + \beta \int \frac{e^{-\eta^2}}{\eta}\, d\eta \qquad (6)$$

α , β are integration constants. For bounded solution at $r \longrightarrow \infty$

$(\eta = \infty)$ $c(\infty,t) = 0$ $\therefore$ $\alpha = 0$

Also $c(0,\infty) = c_0$ $(\eta = 0)$ and $c(1,t) = c_0$
Applying these boundary conditions, we obtain

$$c(r,t) = c_0 \left(1 - \frac{\int_{\frac{1}{2\sqrt{Dt}}}^{\frac{r}{2\sqrt{Dt}}} \frac{e^{-\eta^2}}{\eta} d\eta}{\int_0^{\frac{1}{2\sqrt{Dt}}} \frac{e^{-\eta^2}}{\eta} d\eta} \right) \qquad (7)$$

We now try to obtain the solution by Laplace transforms

$$s\hat{c} = D\left[\frac{d^2}{dr^2}\hat{c} + \frac{1}{r}\frac{d}{dr}\hat{c}\right]$$

i.e.

$$\frac{d^2}{dr^2}\hat{c} + \frac{1}{r}\frac{d}{dr}\hat{c} - \frac{s}{D}\hat{c} = 0$$

This is the modified Bessel equation of zero order. Bounded solution in the interior of unit circle is

$$\hat{c}\,(r,s) = \beta_1 I_0\left(\sqrt{\frac{s}{D}}\, r\right), \; r < 1$$

I_0 is the modified Bessel function of the first kind of order zero, and bounded solutions at ∞ (Gupta, 1978b).

$$\hat{c}\,(r,s) = \beta_2 K_0\left(\sqrt{\frac{s}{D}}\, r\right), \; r > 1$$

K_0 is the modified Bessel function of the second kind of order zero .

In both cases

$c(1,s) = \frac{1}{s}$ $\therefore$ $\beta_1 = \frac{1}{s}\frac{1}{I_0\sqrt{\frac{s}{D}}}$ and $\beta_2 = \frac{1}{s}\frac{1}{K_0\sqrt{\frac{s}{D}}}$. Giving

$$c(r,t) = L^{-1}\left\{\frac{c_0}{s}\;\frac{I_0\left(\sqrt{\frac{s}{D}}\, r\right)}{I_0\left(\sqrt{\frac{s}{D}}\right)}\right\}, r < 1 \qquad (8a)$$

And

$$c(r,t) = L^{-1}\left\{\frac{c_0}{s}\;\frac{K_0\left(\sqrt{\frac{s}{D}}\, r\right)}{K_0\left(\sqrt{\frac{s}{D}}\right)}\right\}, r > 1 \qquad (8b)$$

And both two inverses are equal to

$$c(r,t) = c_0 \left(1 - \frac{\int_{\frac{1}{2\sqrt{Dt}}}^{\frac{r}{2\sqrt{Dt}}} \frac{e^{-\eta^2}}{\eta} d\eta}{\int_0^{\frac{1}{2\sqrt{Dt}}} \frac{e^{-\eta^2}}{\eta} d\eta} \right), \text{ for all } r$$

For the spherically symmetric case (Beyer, 1991a).

This has the solution for $\eta = \frac{r}{2\sqrt{Dt}}$

Given by

$$c\left(\eta = \frac{r}{2\sqrt{Dt}}\right) = \gamma + \delta \int \frac{e^{-\eta^2}}{\eta^2}\, d\eta$$

γ and δ are integration constants

We can easily conclude for similar initial and boundary conditions that

$$c(r,t) = c_0 \left(1 - \frac{\int_{\frac{1}{2\sqrt{Dt}}}^{\frac{r}{2\sqrt{Dt}}} \frac{e^{-\eta^2}}{\eta^2} d\eta}{\int_0^{\frac{1}{2\sqrt{Dt}}} \frac{e^{-\eta^2}}{\eta^2} d\eta} \right) \qquad (9)$$

And transform gives

$$s\hat{c} = D\left[\frac{d^2\hat{c}}{dr^2} + \frac{2}{r}\frac{d\,\hat{c}}{dr}\right]$$

i.e.

$$r^2\frac{d^2\hat{c}}{dr^2} + 2r\frac{d\,\hat{c}}{dr} - \frac{s}{D}\, r^2\,\hat{c} = 0$$

This is modified spherical Bessel equation of zero order (Beyer, 1991b) which possess a solution in terms of the spherical Bessel functions (Beyer, 1991b).

$$\hat{c}(r,s) = \delta_1 \sqrt{\frac{\pi}{2}\frac{D}{s\,r}}\; J_{1/2}\left(\sqrt{\frac{s}{D}}\; r\right), \quad r<1$$

$J_{1/2}$ is the Bessel function of first kind of order $\frac{1}{2}$, and

$$\hat{c}(r,s) = \delta_2 \sqrt{\frac{\pi}{2}\frac{D}{s\,r}}\; Y_{1/2}\left(\sqrt{\frac{s}{D}}\; r\right), \quad r>1$$

$Y_{1/2}$ is the Bessel function of the second kind of order $\frac{1}{2}$.

Applying conditions, we conclude that

$$L^{-1}\left\{\frac{c_0}{s}\frac{J_{1/2}\left(\sqrt{\frac{s}{D}}\,r\right)}{J_{1/2}\left(\sqrt{\frac{s}{D}}\right)}\right\},\quad r < 1$$

$$L^{-1}\left\{\frac{c_0}{s}\frac{Y_{1/2}\left(\sqrt{\frac{s}{D}}\,r\right)}{Y_{1/2}\left(\sqrt{\frac{s}{D}}\right)}\right\},\quad r > 1$$

Are given by the expression (9)

CONCLUSION

In this analysis we succeeded in determining some difficult inverse Laplace transforms involving special functions by employing similarity variable solutions. Other methods of solutions can be applied for comparison; such as the Fourier transform methods. If the involved integrals could be obtained; inverses can be determined explicitly. Otherwise these inversions are expressed as unevaluated integrals.

REFERENCES

Beyer, WH. 1991[a]. Standard Mathematical Tables and formulae. CRC Press, London. pp329.

Beyer, WH. 1991[b]. Standard Mathematical Tables and formulae. CRC Press, London. pp358.

DiBenedetto, E. 2010. Partial Differential Equations, (2[nd] edi.). Springer, NY, USA. 172-176.

Evans, LC. 2010. Partial Differential Equations, (2[nd] edi.). American Mathematical Society. 176-187.

Gupta, BD. 1978[a]. Mathematical Physics, Vikas publishing house, India. 856-918.

Gupta, BD. 1978[b]. Mathematical Physics. Vikas Publishing House, India. 920-921.

Harold, L. 1997. Partial Differential Equations. American Mathematical Society. 5:51-61.

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