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Short Communication

FORCED VIBRATIONS OF CANTILEVER BEAMS

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ABSTRACT

In this paper, the problem of forced vibrations of beams is investigated. The initial-boundary value problem is formulated. The cantilever beam-fixed at one end, and free in the other- is given prescribed initial displacement and velocity. The natural frequencies and the normal modes are obtained as the eigenvalues and the eigenvectors of the corresponding eigenvalue problem.

Keywords: Cantilever, vibration, natural frequency, normal modes, eigenvalues, eigenvectors.

INTRODUCTION

Structures in general are frequently subject to vibrations, traffic, earthquakes beside explosions are common sources of vibrations. Beams of all kinds are important structural elements and therefore, subject to such vibrations. Forced vibrations arise when external loads are applied to the inertia excited vibrating elements. These external loads can be static-as in the case of dead loads- or dynamic, as those resulting from reciprocating machines producing periodic loads – if these external loads are absent, the vibrations are those referred to as free vibrations. Vibrations in general result due to mutual exchange between kinetic and elastic potential energies of the system.

Rezaee and Hassannejad (2011) has provided a new approach to free vibration analysis based on the mechanical energy balance method, and their paper (2010); on free vibration analysis of simply supported beam using perturbation method. Shavezipur and Hashemi (2009) discussed the free vibration of triply coupled centrifugally stiffened non-uniform beams using the dynamic finite element method. The problem of the influence of shear deformation and rotary inertia on non-linear free vibration of a beam with pinned ends is investigated by Foda (1999). Ghayesh and Balar (2008) worked on non-linear parametric vibration and stability of axially moving visco-elastic Rayleigh beams. Sun and Huang (2001) published a paper on vibration suppression of laminated composite beams with a piezo-electric damping layer. Also, the problem of transverse vibrations of an Euler-Bernoulli uniform beam carrying two particles in-span is studied by Naguleswaran (2001). Au *et al*. (1999) investigated the vibration and stability of non-uniform beams with abrupt changes of cross-section using c^1 modified beam vibration functions.

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Formulation of the problem

Consider the cantilever beam shown in figure 1. The axis of the beam is along the x −axis, the cantilever is fixed at $x = 0$ and extends to the free end at $x = l$. The beam is subjected to transverse uniform load with intensity q per unit length. The modulus of elasticity of the beam material is E and the moment of inertia about the neutral axis is I. Upon vibration, the mass of the beam of mass density ρper unit length- will oscillate with acceleration $\frac{\partial^2 y}{\partial t^2}$; y is the transverse deflection of the beam and t is the time. Accordingly, the equation of transverse motion of the beam will be

$$EI\,\frac{\partial^2 y}{\partial x^2} = M(x,t) \qquad (1)$$

M is the bending moment and the dependence of the moment on time comes through the oscillating mass. Evaluating the bending moment at an arbitrary section at x distance from the fixed end; equation (1) will become

$$EI\,\frac{\partial^2 y}{\partial x^2} = \frac{1}{2}(l-x)^2\left(q + \rho\,\frac{\partial^2 y}{\partial t^2}\right) \qquad (2)$$

Using l to non-dimensionalize distance and denoting $\eta = \frac{1}{\sqrt{2}}(1-x)$, equation (2) reduce to

$$\frac{\partial^2 y}{\partial \eta^2} = 2\,\eta^2\left(Q + \frac{\partial^2 y}{\partial \tau^2}\right) \qquad (3)$$

Where $Q = \frac{q\,l^3}{E\,I}$ and $\tau = \sqrt{\frac{E\,I}{\rho\,l^4}}\;t$

We seek a solution of (3) in the form

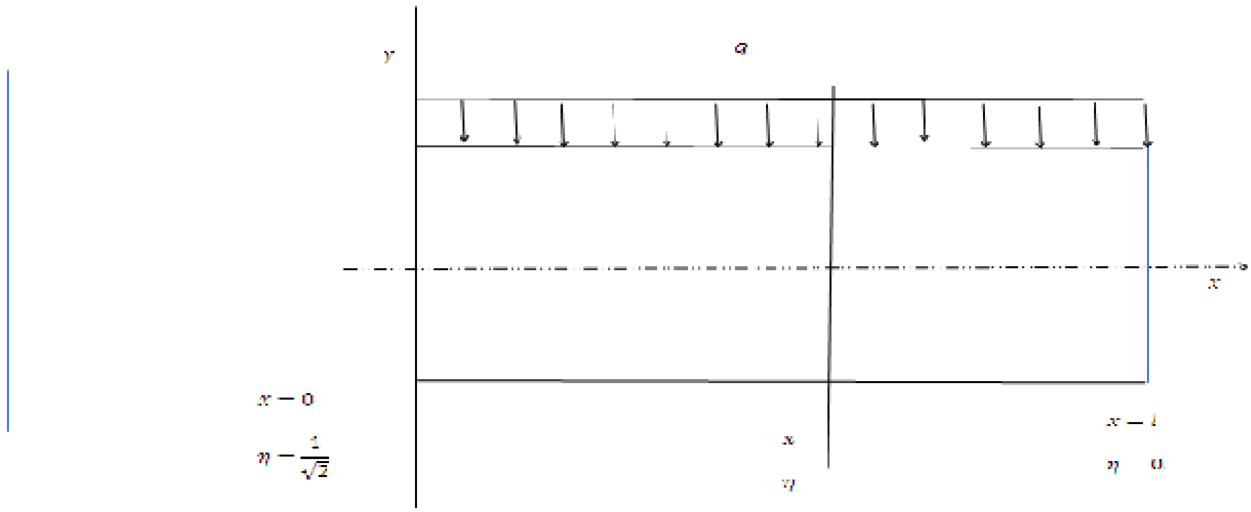

Fig. 1. Schematic of the cantilever beam.

$$y(\eta,\tau) = y_q(\eta) + y_p(\eta,\tau) \tag{4}$$

This superposition is possible because the equation is linear and because q is not a function of time. We have

$$\frac{\partial^2 y_q}{\partial \eta^2} = 2\,\eta^2 Q \tag{5}$$

and

$$\frac{\partial^2 y_p}{\partial \tau^2} = \frac{1}{2\,\eta^2}\frac{\partial^2 y_p}{\partial \eta^2} \tag{6}$$

The initial and boundary conditions will be

$$y(\eta,0) = H(\eta);\ \frac{\partial y(\eta,0)}{\partial \tau} = G(\eta) \tag{7}$$

The functions H and G are prescribed. The end conditions will be

$$y_q\left(\eta = \frac{1}{\sqrt{2}}\right) = \frac{\partial y_q\left(\eta = \frac{1}{\sqrt{2}}\right)}{\partial \eta} = 0 \tag{8}$$

$$y_p(\eta = 0) = y_p\left(\eta = \frac{1}{\sqrt{2}}\right) = \frac{\partial^2 y_p(\eta = 0)}{\partial \eta^2} = 0 \tag{9}$$

Here we assigned the value $H(0), G(0)$ to the static y_q part.

Integrating equation (5) together with the boundary condition (8) gives:

$y_q = Q\left[\frac{1}{6}\eta^4 - \frac{1}{3\sqrt{2}}\eta + \frac{1}{8}\right]$ giving tip deflection

$$\frac{Q}{8} = \frac{q\,l^3}{8\,E\,I} \tag{10}$$

The dynamic part of the solution y_p is sought in the form:

$$y_p(\eta,\tau) = \sum_n A_n e^{i\Omega_n \tau} R_n(\eta) \tag{11}$$

Substituting the solution (11) in equation (6) gives

$$R_n'' + 2\,\Omega_n^2\,\eta^2 R_n = 0 \tag{12}$$

The values of Ω_n are the natural frequencies and R_n are the resulting normal modes. We postulate that Ω_n are real since the system is conservative and no energy dissipation is considered as there is no damping. Accordingly R_n are expected to be real. No closed form solution of (12) can be evaluated. With trivial boundary conditions numerical solution will gives the values of the eigenvalues Ω_n and the eigenvectors R_n.

With real eigenvalues and eigenvectors the solution (11) should take the form

$$y_p(\eta,\tau) = \sum_n (A_n \cos\Omega_n \tau + B_n \sin\Omega_n \tau) R_n(\eta) \tag{13}$$

The coefficients A_n and B_n are estimated for the initial conditions from:

$$y(\eta,0) = H(\eta) = \sum_{n=1}^{m-1} A_n\,R_n(\eta) \tag{14a}$$

and

$$\frac{\partial y(\eta,0)}{\partial \tau} = G(\eta) = \sum_{n=1}^{m-1} \Omega_n B_n R_n(\eta) \tag{14b}$$

This is done numerically; m is the number of subdivisions of the interval $\left[0, \frac{1}{\sqrt{2}}\right]$

Numerical work

Equation (12) with the boundary conditions (9) is Sturm-Liouville problem, the eigenvalues and eigen-functions (eigenvectors) can be obtained numerically, the interval $[0,\frac{1}{\sqrt{2}}]$ is divided into m subdivisions, denote the solution at $\eta = \eta_n$ by U_n, equation (9) is written at $\eta = \eta_n$ and 2nd derivative R'' is approximated by $\frac{1}{h^2}(R_{n-1} - 2R_n + R_n)$ where $h = \frac{1}{m\sqrt{2}}$. Then for $n = 1,2, \ldots . m-1$, we obtain the system

$$\begin{pmatrix} 2-\lambda & -1 & 0 & & & & \\ -\frac{1}{2^2} & \frac{2}{2^2}-\lambda & -\frac{1}{2^2} & & 0 & & \\ \vdots & \vdots & \vdots & & & & \\ & & & \vdots\ \vdots & & & \\ & & & & \vdots & \vdots & \vdots \\ 0 & & & & \frac{-1}{2^{m-2}} & \frac{2}{2^{m-2}}-\lambda & \frac{-1}{2^{m-2}} \\ & & & & 0 & \frac{-1}{2^{m-1}} & \frac{2}{2^{m-1}}-\lambda \end{pmatrix} \begin{pmatrix} R_1 \\ R_2 \\ \vdots \\ \vdots \\ R_{m-2} \\ R_{m-1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \\ 0 \end{pmatrix} \tag{15}$$

Where $\lambda = \frac{\Omega^2}{2m^4}$

Due to Gerschgorin (1994)'s theorems, the eigenvalues are real and lie in the interval $\left[\frac{1}{2^{m-1}}, 3\right]$. If the eigenvalues $\lambda_1 > \lambda_2, \ldots\ldots > \lambda_{m-1}$, the corresponding eigenvectors are R_1, R_2,……………., R_{m-1} and the natural frequencies are

$$\Omega_n = m^2\sqrt{2\lambda_n},\ n = 1,2, \ldots, m-1.$$

If we take m=5, by the help of Mathcad plus 5, the eigenvalues and the eigenvectors are obtained as listed in table 1. Also, if $H(\eta) = 0.1\,\eta(1-\sqrt{2}\eta)$ and $G(\eta) = 0.2\,\eta(1-\sqrt{2}\eta)$, the $A's$ and $B's$ in equations (14) are computed.

CONCLUSION

The boundary conditions at ends should be satisfied with the static and dynamic parts of the solutions independently since they are satisfied at all times. Also, we notice that the success of the eigenvalue scheme used to obtain the eigenvalues-the natural frequencies-and the eigenvectors-the normal modes –depend on the zero value assigned to the tip of the cantilever in the dynamic part of the solution. This is admitted if the prescribed tip deflection is assigned to the static q part of the solution. The two functions $H(\eta)$ and $G(\eta)$ chosen for demonstration are zero at the tip this enables decoupling of the two component solutions y_q and y_p. The eigen-modes will be affected by y_q as a coordinate function of the expansion of arbitrary $H(\eta)$ and $G(\eta)$ which do not vanish tip.

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Table 1. Numerical Results.

Values of η	$\eta_0 = 0$	$\eta_1 = \frac{1}{5\sqrt{2}}$	$\eta_2 = \frac{2}{5\sqrt{2}}$	$\eta_3 = \frac{3}{5\sqrt{2}}$	$\eta_4 = \frac{4}{5\sqrt{2}}$	$\eta_5 = \frac{1}{\sqrt{2}}$
Values of Ω		Ω_1=76.1	Ω_2=16.4	Ω_3=6.5	Ω_4=1.7	
Values of R_1	0	0.9885	0.5011	0.3568	0.2348	0
Values of R_2	0	-0.1508	0.7701	0.6479	0.4975	0
Values of R_3	0	0.0087	-0.3880	0.4621	0.6461	0
Values of R_4	0	-0.00076	0.0717	-0.4892	0.51970	0
$H(\eta) = 0.1\eta(1-\sqrt{2}\eta), G(\eta) = 0.2\eta(1-\sqrt{2}\eta)$						
Values of A_n	0	0.00178	0.00273	0.00431	0.025	0
Values of B_n	0	0.000047	0.000333	0.001325	0.029	0

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