SENRA Academic Publishers, Burnaby, British Columbia
Vol. 6, No. 1, pp. 1897-1900, Feb 2012
Online ISSN: 1920-3853; Print ISSN: 1715-9997

Canadian Journal of pure & applied sciences

Short Communication

CONFORMAL MAPPING TECHNIQUE FOR STUDYING FLUID FLOW IN CONTRACTION GEOMETRY

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ABSTRACT

In this paper, fluid flow over contraction geometry with a moving lower edge is studied by using conformal mapping techniques. Using elliptic integral is presented. The resulting streamlines are very much convenient and acceptable.

2000 AMS: Subject classification 65M12**,** 65M50, 76-08

Keywords: Conformal mapping, elliptic integral, streamlines, lower edge.

INTRODUCTION

In many engineering applications, lubrication, channel flows, pipe flows etc, the contraction appears frequently, which makes it necessary to study thoroughly the distribution of the streamlines and its values along the geometry of the flow with different contraction ratios. In this paper we restrict ourselves to domains which can be broken up into a union of smi-infinit rectangles. In particular we consider the contraction geometry which is an infinite channel whose diameter changes abruptly. In this paper we used the conformal mapping for studying the flow of a fluid in contraction geometry by computing its streamlines through transformed it to a flow on rectangle, (Ismail and Allan, 1995). This transformations are determined by using Schwarz-Christoffel (S-C) transformation which its numerical computations have been carried out by different methods in recent years. For example, the conformal mapping of unit disk onto a prescribed polygon has been studied successfully by Davis (1979). The mapping from a straight channel to a channel of arbitrary shape and periodic channel has been investigated by Floryan (1985). We define a Laplace's equation with mixed boundary conditions on contraction geometry and transform the whole problem onto a definite rectangle by means of the upper half plane .we use the fact that: if the fluid is following Laplace's equation, then under (S-C) transformation its form is unaltered. But if the flow is obeying Poisson's equation, the change is only in the source function (Dorr, 1970).

Schwarz-Christoffel transformation.
Consider the Schwarz-Christoffel integral

$$z=c\int_0^t (t-a_1)^{\alpha_1-1}(t-a_2)^{\alpha_2-1}\dots(t-a_{N-1})^{\alpha_{N-1}-1}\,dt+c_1 \qquad (1)$$

$$=c\int_0^t \prod_{j=1}^{N-1}(t-a_j)^{\alpha_j-1}\,dt+c_1 \qquad (2)$$

which maps the upper half of the t-plane to interior of open polygon in the z-plane. There are a total of $2N+2$ parameters

$\alpha_1,\alpha_2,\dots,\alpha_N$, $a_1,a_2,\dots,a_N$, c and c_1 where $a_1,a_2,\dots,a_N$ are points on the real axis whose images are to be vertices of the polygon and the inequality relations:

$$a_1 < a_2 < \dots < a_{N-1} < a_N \qquad (3)$$

are taken as the constraints of the system. The parameters $\alpha_1\pi$, $\alpha_2\pi,\dots,\alpha_N\pi$ denote interior angles of the polygon with N vertices such that

$$\alpha_1\pi + \alpha_2\pi + \dots + \alpha_N\pi = (N-2)\pi \qquad (4)$$

It is noted that the Nth parameters α_N and a_N don't appear explicity in the integral since the last one of the vertices of the polygonal region is mapped to a point at infinity on the real axis of the ζ-plane. Vertices of the polygon on the z-plane and these corresponding mapping points along the real axis on the ζ-plane satisfy the relation

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$$z_k = c\int_0^{a_k} \prod_{j=1}^{N-1} (t-a_j)^{\alpha_j - 1} dt + c_1, k = 1,2,...,N \tag{5}$$

Mathematical formulation of the problem

The problem mathematically amounts to the solution of Laplace's equation $\nabla^2 \psi = 0$ with mixed boundary conditions as given in figure 1 below

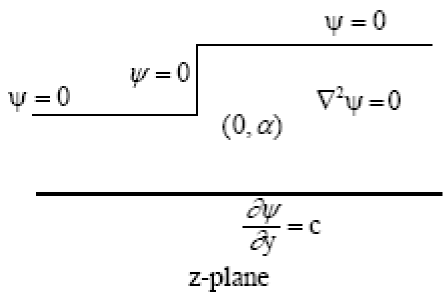

Fig. 1. Contraction geometry and boundary conditions.

where ψ represent the stream function of the fluid that is moving from right to left through some contraction of specified width. We here assume that the fluid is irrotational non-viscous, bearing in mind that the same method of solution which be described here. Below can be used also in the case of rotational non-viscous flow.

Transformation from the upper half plane to the contraction geometry

The contraction geometry, Ω of figure 1 can be mapped conformally onto the rectangle , Ω'' defined by

$$\Omega'' = \{(u,v): -a < u < a,\ 0 < v < b\}.$$

This transformation is performed in two stages:

1- Let $z = x + iy$ (contraction geometry) defined on the z-plane and $t = \xi + i\eta$ (upper-half), then Ω can be mapped conformally onto the upper half plane by means of the transformation

$$\frac{dz}{dt} = \frac{-1}{\pi t}\left[\frac{t-1}{t-\left(\frac{1}{\alpha^2}\right)}\right]^{\frac{1}{2}} \tag{6}$$

where α is contraction ratio. If we write $w = u + iv$ then Ω' is mapped conformally onto Ω'' y means of the transformation

$$\frac{dt}{dw} = -\pi t \tag{7}$$

2-Given a point w w in the region Ω'' it is a simple task to write down the point in z in Ω to which it corresponds

$$z = i - \frac{1}{\pi}\left\{\log\left[\frac{\sqrt{w-1}+\sqrt{w-\beta}}{\sqrt{w-1}-\sqrt{w-\beta}}\right] - \alpha\log\left[\frac{\sqrt{w-1}+\alpha\sqrt{w-\beta}}{\sqrt{w-1}-\alpha\sqrt{w-\beta}}\right]\right\} \tag{8}$$

where $t = -\exp(-\pi w)$ and $\beta = \frac{1}{\alpha^2}$.

One cannot, in a straightforward manner, write down the analytic solution in Ω. One would need to find analytically $w(z)$ given above. What we have done is to take a point w in Ω'' n which the analytic solution is known, then find z in contraction.

Finding the points of mapping from the upper half-plane to rectangle.

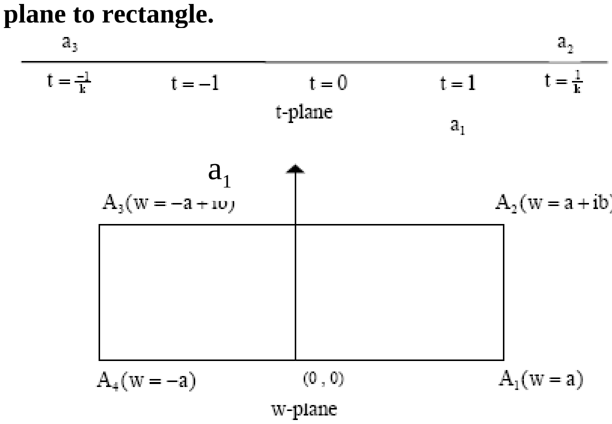

Fig. 2. Mapping from upper plane to rectangle.

By using (S-C) transformation, the function which defines a conformal mapping of the upper half plane onto a given rectangle may be represented in the form

$$w = c\int_{t_0}^{t} (t-1)^{\frac{1}{2}-1}\left(t-\frac{1}{k}\right)^{\frac{1}{2}-1}\left(t+\frac{1}{k}\right)^{\frac{1}{2}-1}(t+1)^{\frac{1}{2}-1} dt + c_1 \tag{9}$$

$$w = c\int_{t_0}^{t} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} + c_1 \tag{10}$$

by using the correspondence points, we get $c_1 = 0$ and

$$a = c\int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}, 0 < k < 1 \tag{11}$$

The integral on the right is the so-called complete elliptic integral of the first kind which does not have a closed form. In order to be able to perform the mapping, we have to find the value of the above integral at each chosen point of the upper half-plane. The above integral written in the form

$$F(k,\phi)=\int_0^{\phi}\frac{d\phi}{\sqrt{(1-k^2\sin^2\phi)}} \tag{12}$$

By using Landen's transformation,

$$\tan\phi=\frac{\sin 2\phi_1}{k+\cos 2\phi_1} \quad or\; k\sin\phi=\sin(2\phi_1-\phi) \tag{13}$$

we get

$$F(k,\phi)=\int_0^{\phi}\frac{d\phi}{\sqrt{(1-k^2\sin^2\phi)}}=\frac{2}{1+k}\int_0^{\phi_1}\frac{d\phi_1}{\sqrt{1-k_1^2\sin^2\phi_1}} \quad where,k_1=\frac{2\sqrt{k}}{1+k}. \tag{14}$$

This can be written $\mathrm{F}(\mathrm{k},\phi)=\frac{2}{1+\mathrm{k}}\mathrm{F}(\mathrm{k}_1,\phi_1)$

It is seen that $\mathrm{k}<\mathrm{k}_1<1$. By, successive application of Landen's transformation a sequence of moduli $\mathrm{k}_n, n=1,2,3,\ldots$ is obtained such that $\mathrm{k}<\mathrm{k}_1<\mathrm{k}_2\ldots<1$ and we can prove that $\lim_{n\to\infty}\mathrm{K}_n=1.$ From this it follows that

$$F(k,\phi)=\sqrt{\frac{k_1k_2k_3\ldots}{K}}\int_0^{\phi}\frac{d\theta}{\sqrt{(1-\sin^2\theta)}}=\sqrt{\frac{k_1k_2k_3\ldots}{K}}-\ln\left(\tan\left(\frac{\pi}{4}+\frac{\phi}{2}\right)\right) \tag{15}$$

where

$$\mathrm{K}_1=\frac{2\sqrt{\mathrm{k}}}{1+\mathrm{k}},\quad \mathrm{K}_2=\frac{2\sqrt{\mathrm{k}_1}}{1+\mathrm{k}_1},\quad \mathrm{K}_3=\frac{2\sqrt{\mathrm{k}_2}}{1+\mathrm{k}2}, \tag{16}$$

and $\phi=\lim_{n\to\infty}\phi_n$

By using this result, $F(k,\phi)$ can be computed.

Finding the values of ψ over the rectangle

This is equivalent to solving Laplace's equation with mixed boundary conditions as in figure 3.

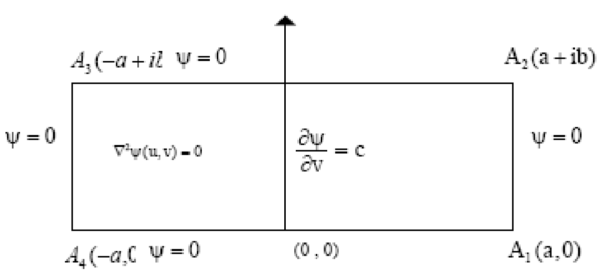

Fig. 3. Rectangle geometry and boundary conditions.

The final solution of this problem on the above rectangle given by

$$\psi(u,v)=\sum_{n=odd}\frac{8ac}{n^2\pi^2\cosh\left(\frac{n\pi b}{2a}\right)}\sin\frac{n\pi(u+a)}{2a}\sinh\frac{n\pi v}{2a} \tag{17}$$

RESULTS AND DISCUSSION

We have designed a program written in FORTRAN to calculate the streamlines of the solution ψ of the entire problem at every points of a rectangle. The results show that the method described in this paper gives accurate results in the whole domain of contraction geometry. The streamlines of Laplace's equation on the rectangle is given in figure 4 and the solution over the contraction geometry shown in figures 5,6 at different contraction ratios which provides the streamlines of some contraction with a base that is moving with a constant velocity. This results is very acceptable compared to those obtained by Phillips and Davies (1988) and Chuang and Hsiung (1993).

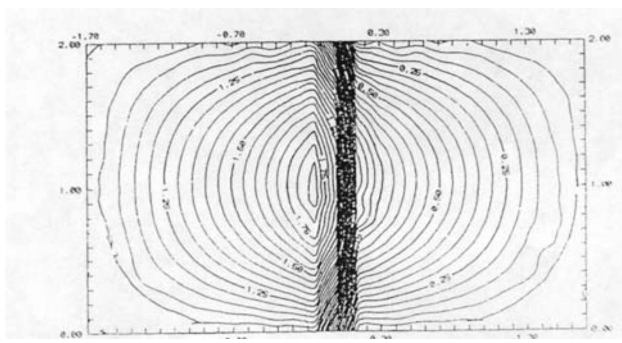

Fig. 4. Contours of ψ in the rectangle.

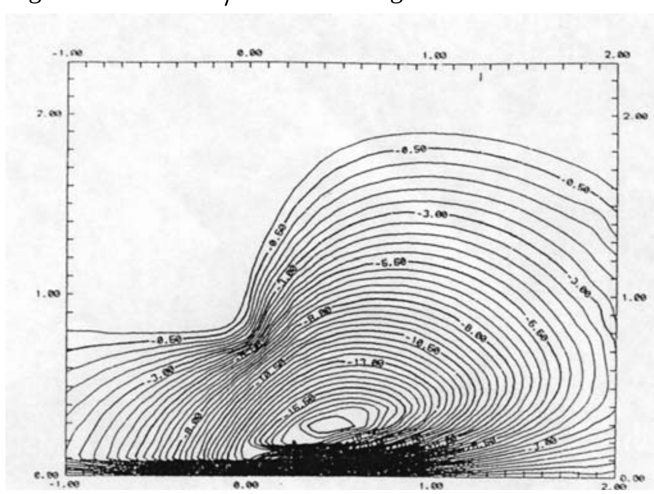

Fig. 5. Contours of ψ in the contraction with $\alpha=.5$

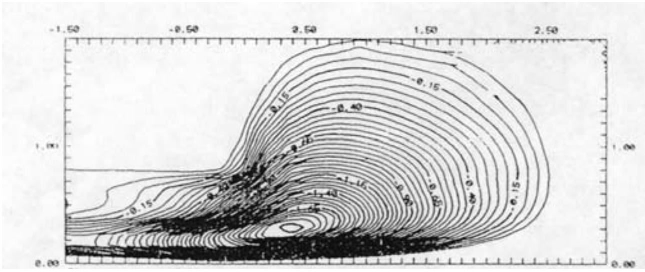

Fig. 6. Contours of ψ in the contraction with $\alpha = .25$

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Received: June 7, 2011; Revised: Dec 4, 2011; Accepted: Dec 5, 2011