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COMMON FIXED POINT THEOREMS FOR OCCASIONALLY WEAKLY COMPATIBL MAPS IN FUZZY METRIC SPACES

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ABSTRACT

The intent of this paper is to introduce the notion of occasionally weakly compatible (owc) maps and prove common fixed point theorems for single and set valued maps without considering the completeness of the space and continuity of maps in fuzzy metric space. Our results extend, generalize and unify several results existing in the literature.

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INTRODUCTION

It was a turning point in the development of mathematics when Zadeh (1965) introduced the concept of fuzzy set. This laid the foundation of fuzzy mathematics. Consequently, the last three decades were very productive for fuzzy mathematics. Several authors like Deng (1982), Erceg (1979), Kaleva and Seikkala (1984) and Kramosil and Michalek (1975) have introduced the concept of fuzzy metric space in different ways. The concepts of weak commutatively, compatibility, non-compatibility and weak compatibility were frequently used to prove fixed point theorems for single and set valued maps satisfying certain conditions in different spaces.

Al-Thagafi and Shazad (2008) weakened the notion of weakly compatible maps by introducing a new concept of occasionally weakly compatible (owc) maps. This concept is more general among all the commutativity concepts and has opened a new venue for many mathematicians. This newly defined concept has also fascinated many authors like Alamgir and Sumitra (2010), Bouhadjera and Thobie (2008, 2009), Abbas and Rhoades (2007), Gairala and Rawat (2009), and Chandra and Bhatt (2009).

The main purpose of the present paper is to introduce the concept of occasionally weakly compatible (owc) maps in fuzzy metric space and to prove common fixed point theorems for single and set valued maps under strict contractive condition.

Our improvements in this paper are five-fold as:

(i) Relaxed the continuity of maps completely
(ii) Completeness of the space removed
(iii) Minimal type contractive condition used
(iv) The condition $\lim_{t\to\infty} M(x,y,t)=1$ not used
(v) Weakened the concept of compatibility by a more general concept of occasionally weak compatible (owc) maps .

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We first give some preliminaries and definitions.

1. Preliminaries

Definition 1.1. A binary operation $*:[0,1]\times[0,1]\to[0,1]$ is continuous $t-norm$ if $*$ is satisfying the following conditions:

(i) $*$ is commutative and associative
(ii) $*$ is continuous
(iii) $a*1=a$ $for\ all$ $a\in[0,1]$
(iv) $a*b\le c*d$ $whenever$ $a\le c$ and $b\le d$, $a,b,c,d\in[0,1]$.

Definition 1.2. A triplet $(X,M,*)$ is said to be a fuzzy metric space if X is an arbitrary set, $*$ is a continuous $t-norm$ and M is a fuzzy set on $X^2\times(0,\infty)$ satisfying the following;

(FM-1) $M(x,y,t)>0$
(FM-2) $M(x,y,t)=1$ $if\ and\ only\ if$ $x=y.$
(FM-3) $M(x,y,t)=M(y,x,t)$
(FM-4) $M(x,y,t)*M(y,z,s)\le M(x,z,t+s)$
(FM-5) $M(x,y,\bullet):(0,\infty)\to(0,1]$ $is\ continuous.$

Note that $M(x,y,t)$ can be thought of as the degree of nearness between x and y with respect to t .

Example 1. (Induced fuzzy metric space) Let (X,d) be a metric space, denote $a*b=ab$ for all $a,b\in[0,1]$ *and let* M_d *be* fuzzy set on $X^2\times(0,\infty)$ defined as follows;

$$M_d(x,y,t)=\frac{t}{t+d(x,y)}.$$

Then $(X,M_d,*)$ is a fuzzy metric space. We call this fuzzy metric induced by a metric d.

Example 2 Let $X=N$. *Define* $a*b=\max\{0,a+b-1\}$ *for all* $a,b\in[0,1]$ and let M be a fuzzy set on $X^2\times(0,\infty)$ as follows;

$$M(x,y,t)=\begin{Bmatrix}\frac{x}{y}\ if\ x\le y\\ \frac{y}{x}\ if\ y\le x\end{Bmatrix}\ for\ all\ x,y\in X.\ Then\ (X,M,*)$$

is a fuzzy metric space.

Note that in the above example, there exists no metric d on X, satisfying

$$M(x,y,t)=\frac{t}{t+d(x,y)},\ where(X,M,*)\text{ is}$$

defined in above example. Also note that the above function M is not a fuzzy metric with the $t-norm$ defined as $a*b=\min\{a,b\}$.

Through out the paper X will represent the fuzzy metric space $(X,M,*)$ and $CB(X)$, the set of all non-empty closed and bounded sub-sets of X. For $A,B\in CB(X)$ and for every $t>0$, denote

$$H(A,B,t)=\sup\{M(a,b,t;a\in A,b\in B)\}$$

and

$$\delta_M(A,B,t)=Inf\ \{M(a,b,t);a\in A,b\in B\}.$$

If A consists of a single point a, we write $\delta_M(A,B,t)=\delta_M(a,B,t)$. If B also consists of a single point b, we write $\delta_M(A,B,t)=M(a,b,t)$.

It follows immediately from definition that

$$\delta_M(A,B,t)=\delta_M(B,A,t)\ge 0$$

$$\delta_M(A,B,t)=1\Leftrightarrow A=B=\{a\}\text{ for all } A,B\in CB(X)$$

Definition 1.3. A point $x\in X$ is called a coincidence point (respective fixed point) of $A:X\to X$, $B:X\to CB(X)$ if $Ax\in Bx$ (respective $x=Ax\in Bx$)

Definition 1.4. Maps $A:X\to X$ and $B:X\to CB(X)$ are said to be compatible if $ABx\in CB(X)$ for all $x\in X$ and $lin_{n\to\infty}H(ABx_n,BAx_n,t)=1$ whenever $\{x_n\}$ is a sequence in X such that $Bx_n\to M\in CB(x)$ and $Ax_n\to x\in M$.

Definition 1.5. Maps $A:X\to X$ and $B:X\to CB(X)$ are said to be weakly compatible if they commute at coincidence points. i.e., if $ABx=BAx$, whenever $Ax\in Bx$.

Definition 1.6. Maps $A:X\to X$ and $B:X\to CB(X)$ are said to be occasionally weakly compatible (owc) if there exists some point $x\in X$ such that $Ax\in Bx$ and $ABx\subseteq BAx$.

Clearly weakly compatible maps are occasionally weakly compatible (owc). However, the converse is not true in general as shown in the following example.

Example 3. Let $X=[0,\infty)$ with $a*b=\min\{a,b\}$ for all $a,b\in[0,1]$ and

$$M(x,y,t)=\frac{t}{t+d(x,y)}\text{ for all } t>0.$$

Define the maps $A:X\to X$ and $B:X\to CB(X)$ by setting

$$Ax=\begin{Bmatrix}0,\ 0\le x<1\\ x+1,\ 1\le x<\infty\end{Bmatrix},$$

$$Bx=\begin{Bmatrix}\{0\},\ 0\le x<1\\ [1,x+2],\ 1\le x<\infty\end{Bmatrix}$$

Here '1' is a coincidence point of A and B but A and B are not weakly compatible as

$$BA(1)=[1,4]\ne AB(1)=[2,4].$$

Hence A and B are not compatible.

But A and B are occasionally weakly compatible (owc) as A and B are weakly compatible at $x=0$ as $A(0)\in B(0)$ and $AB(0)\subseteq BA(0)$. i.e., $A\{0\}=0\subseteq B(0)=\{0\}$

2. RESULTS AND DISCUSSION

Now, we prove the following result.

Theorem 1. Let $(X,M,*)$ be a fuzzy metric space with $t*t=t$. Let $A,B:X\to X$ and $S,T:X\to CB(X)$ be single and set valued mappings respectively such that the maps (A,S) and (B,T) are (owc) and satisfy the inequality

(1.1)
$$\delta_M(Sx,Ty,t)\ge\phi\left[\min\left\{\begin{matrix}M(Ax,By,t),H(Ax,Sx,t),\\ H(By,Ty,t),H(Ax,Ty,\alpha t)*H(By,Sx,(2-\alpha)t)\end{matrix}\right\}\right]$$

for every $x,y\in X$, $t>0$, $\alpha\in(0,2)$, where $\phi:[0,1]\to[0,1]$ is continuous function such that $\phi(s)>s$ for each $0<s<1$. Then A, B, S and T have unique common fixed point in X.

Proof. Since the pairs (A,S) and (B,T) are occasionally weakly compatible (owc) maps , therefore, there exist two elements u,v in X such that $Au\in Su$, $ASu\subseteq SAu$ and $Bv\in Tv$, $BTv\subseteq TBv$.

First we prove that $Au=Bv$. As $Au\in Su$, $Bv\in Tv$, so, $M(Au,Bv,t)\ge\delta_M(Su,Tv,t)$ and $M(Bv,Su,t)\ge\delta_M(Tv,Su,t)$

Suppose that $Au\ne Bv$, then $\delta_M(Su,Tv)<1$. Using (1.1) for $x=u,y=v$

$$\delta_M(Su,Tv,t)\ge\phi\left[\min\left\{\begin{matrix}M(Au,Bv,t),H(Au,Su,t),\\ H(Bv,Tv,t),H(Au,Tv,\alpha t)*H(Bv,Su,(2-\alpha)t)\end{matrix}\right\}\right]$$

Since $*$ is continuous , letting $\alpha\to 1$, we get

$$\delta_M(Su,Tv,t)\ge\phi\left[\min\left\{\begin{matrix}M(Au,Bv,t),H(Au,Su,t),\\ H(Bv,Tv,t),H(Au,Tv,t)*H(Bv,Su,t)\end{matrix}\right\}\right]$$

$$\ge\phi\left[\min\{M(Au,Bv,t),1,1,M(Au,Tv,t)*M(Bv,Su,t)\}\right]$$

$$\ge\phi\left[\min\{M(Au,Bv,t),1,1,\delta_M(Su,Tv,t)*\delta_M(Su,Tv,t)\}\right]$$

$$=\phi\left[\min\{M(Au,Bv,t),1,1,\delta_M(Su,Tv,t)\}\right]$$

$=\phi\left[\delta_M(Su,Tv,t)\right]>\delta_M(Su,Tv,t)$, a contradiction.

Therefore, $\delta_M(Su,Tv,t)=1$ and hence $Au=Bv$

Now, we prove that $A^2u=Au$.

Also,

$M(AAu,Au,t)=M(AAu,Bv,t)\ge\delta_M(SAu,Tv,t)$

Suppose $A^2u\ne Au$, then $\delta_M(SAu,Tv,t)<1$

Using (1.1) for $x=Au,y=v$ with $\alpha=1$

$$\delta_M(SAu,Tv,t)\ge\phi\left[\min\left\{\begin{matrix}M(AAu,Bv,t),H(AAu,SAu,t),\\ H(Bv,Tv,t),H(AAu,Tv,t)*H(Bv,SAu,t)\end{matrix}\right\}\right]$$

$$\ge\phi\left[\min\left\{\begin{matrix}M(AAu,Bv,t),1,1,\\ M(AAu,Tv,t)*M(Bv,SAu,t)\end{matrix}\right\}\right]$$

$$\ge\phi\left[\min\left\{\begin{matrix}M(AAu,Bv,t),\delta_M(SAu,Tv,t),1,\\ \delta_M(SAu,Tv,t)*\delta_M(Tv,SAu,t)\end{matrix}\right\}\right]$$

$=\phi\left[\delta_M(SAu,Tv,t)\right]>\delta_M(SAu,Tv,t)$, again a contradiction and hence $\delta_M(SAu,Tv,t)=1$, which yields $AAu=Au=Bv$.

Similarly, we can prove that $B^2v=Bv$

Putting $Au=Bv=z$, then $Az=z=Bz$, $z\in Sz$ and $z\in Tz$.

Therefore, z is a common fixed point of A, B, S and T.

For uniqueness, let $z'\ne z$ be another fixed point of A, B, S and T. Then (1.1) with $\alpha=1$, gives

$$\delta_M(Sz,Tz',t)\ge\phi\left[\min\left\{\begin{matrix}M(Az,Bz',t),H(Az,Sz,t),\\ H(Bz',Tz',t),H(Az,Tz',t)*H(Bz',Sz,t)\end{matrix}\right\}\right]$$

$$\ge\phi\left[\min\left\{\begin{matrix}M(Az,Bz',t),M(Az,Sz,t),\\ M(Bz',Tz',t),M(Az,Tz',t)*M(Bz',Sz,t)\end{matrix}\right\}\right]$$

$$=\phi\left[\min\left\{\begin{matrix}M(z,z',t),M(z,z,t),\\ M(z',z',t),M(z,z',t)*M(z',z,t)\end{matrix}\right\}\right]$$

i.e.,

$$\delta_M(z,z',t)=\delta_M(Sz,Tz',t)>\phi\left[M(z,z',t)\right]>M(z,z',t)>\delta_M(z,z',t)$$

, again a contradiction and hence $z=z'$.

Example 4. Let $(X,M,*)$ be a fuzzy metric space in which $X=[0,\infty)$, $a*b=\min\{a,b\}$ for all $a,b\in[0,1]$ and $M(x,y,t)=\frac{t}{t+d(x,y)}$ $\forall\ t>0$

Define the maps $A,B:X\to X$ and $S,T:X\to CB(X)$ by setting

$$A(x)=\begin{cases} x, & 0\le x\le 1 \\ 2, & 1<x<\infty \end{cases},$$

$$B(x)=\begin{cases} 1, & 0\le x\le 1 \\ x+1, & 1<x<\infty \end{cases}$$

$$S(x)=\begin{cases} \{1\}, & 0\le x\le 1 \\ \{0\}, & 1<x<\infty \end{cases},$$

$$A(x)=\begin{cases} \{x\}, & 0\le x\le 1 \\ [1,x+2], & 1<x<\infty \end{cases}$$

Define $\phi:[0,1]\to[0,1]$ as $\phi(0)=0$, $\phi(1)=1$ and $\phi(s)=\sqrt{s}$ for $0<s<1$, then $\phi(s)>s$.

The pair (A,S) and (B,T) are occasionally weakly compatible (owc) as

$A(1)=1\in S(1)=\{1\}$ and $AS(1)\subseteq SA(1)$. ie.,

$A\{1\}=1\subseteq S(1)=\{1\}$ and

$B(1)=1\in T(1)=[1,3]$ and $BT(1)\subseteq TB(1)$.

Also the contractive condition (1.1) of our theorem is satisfied.

Thus all the conditions of our theorem are satisfied and '1' is the unique common fixed point of A, B, S and T.

Corollary 1. Let $(X,M,*)$ be a fuzzy metric space with $t*t=t$. Let $A,B:X\to X$ and $S,T:X\to CB(X)$ be single and set valued mappings satisfying

(1.2) The pairs (A,S) and (B,T) are owc

(1.3)

$$\delta_M(Sx,Ty,t)\ge\phi\left[\min\left\{\begin{matrix} M(Ax,By,t),H(Ax,Sx,t), \\ H(By,Ty,t),H(Ax,Ty,\alpha t)*H(By,Sx,(2-\alpha)t)\end{matrix}\right\}\right]^h$$

for every $x,y\in X,t>0,\alpha\in(0,2),0<h<1$, where $\phi:[0,1]\to[0,1]$ is continuous function such that $\phi(s)>s$ for each $0<s<1$. Then A, B, S and T have unique common fixed point in X.

Corollary 2. Let $(X,M,*)$ be a fuzzy metric space with $t*t=t$. Let $A,B:X\to X$ and $S,T:X\to CB(X)$ be single and set valued mappings satisfying (1.2) and

(1.4) $\delta_M(Sx,Ty,t)\ge\left[\phi\{H(Ax,By,t)\}\right]^h$

for every $x,y\in X,t>0,0<h<1$, where $\phi:[0,1]\to[0,1]$ is continuous function such that $\phi(s)>s$ for each $0<s<1$. Then A, B, S and T have unique common fixed point in X.

If we put A=B and S=T in theorem 1, we obtain the following result.

Corollary 3. Let $(X,M,*)$ be a fuzzy metric space with $t*t=t$. Let $A:X\to X$ and $S:X\to CB(X)$ be single and set valued mappings respectively such that (A,S) is (owc) and satisfy the inequality

(1.5)

$$\delta_M(Sx,Sy,t)\ge\phi\left[\min\left\{\begin{matrix} M(Ax,Ay,t),H(Ax,Sx,t), \\ H(Ay,Sy,t),H(Ax,Sy,\alpha t)*H(Ay,Sx,(2-\alpha)t)\end{matrix}\right\}\right]$$

for every $x,y\in X,t>0,\alpha\in(0,2)$, where $\phi:[0,1]\to[0,1]$ is continuous function such that $\phi(s)>s$ for each $0<s<1$. Then A and S have unique common fixed point in X.

If we put A=B, we get another corollary.

Corollary 4. Let $(X,M,*)$ be a fuzzy metric space with $t*t=t$. Let $A:X\to X$ and $S,T:X\to CB(X)$ be single and set valued mappings respectively such that the maps (A,S) and (A,T) are (owc) and satisfy the inequality

(1.6)

$$\delta_M(Sx,Ty,t)\ge\phi\left[\min\left\{\begin{matrix} M(Ax,Ay,t),H(Ax,Sx,t), \\ H(Ay,Ty,t),H(Tx,Ty,\alpha t)*H(Ay,Sx,(2-\alpha)t)\end{matrix}\right\}\right]$$

for every $x,y\in X,t>0,\alpha\in(0,2)$, where $\phi:[0,1]\to[0,1]$ is continuous function such that $\phi(s)>s$ for each $0<s<1$. Then A, S and T have unique common fixed point in X.

Now, we prove the projection of theorem 1, from fuzzy metric space to metric space.

Theorem 2. Let (X,d) be a metric space. Let $A,B:X\to X$ and $S,T:X\to CB(X)$ be single and set valued mappings respectively such that the maps (A,S) and (B,T) are (owc) and satisfy the inequality

(1) $$d(Sx,Ty) \le \phi\left[\max\left\{\begin{matrix} d(Ax,By), d(Ax,Sx), \\ d(By,Ty), \frac{1}{2}\{d(Ax,Ty)+d(Ty,Sx)\} \end{matrix}\right\}\right]$$

for every $x, y \in X$, where $\phi : R^+ \to R^+$ be a non-decreasing and $\phi(t) < t$ for every $t > 0$. Then A, B, S and T have unique common fixed point in X.

Proof. The proof follows from theorem 1. Considering the induced fuzzy metric space $(X, M, *)$ where $a * b = \min\{a, b\}$ and $M(x, y, t) = \frac{t}{t + d(x, y)}$

Remark 1. In view of theorem 2, it is clear that some results of Abbas and Rhoades (2007); Al-Thagafi and Shahzad (2008, 2009); Bouhadjera *et al*. (2008); Bouhadjera and Djoudi (2008); Bouhadjera and Thobie (2008); Bouhadjera and Thobie (2009); Chandra and Bhatt (2009) are special cases of our main results in fuzzy metric space.

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