SENRA Academic Publishers, Burnaby, British Columbia
Vol. 5, No. 1, pp. 1429-1432, Feb 2011
Online ISSN: 1920-3853; Print ISSN: 1715-9997

ON ANTI-INVARIANT MAXIMAL SPACELIKE SUBMANIFOLDS OF AN INDEFINITE COMPLEX SPACE FORM

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ABSTRACT

The purpose of this paper is to study the geometry of an n-dimensional anti-invariant maximal spacelike submanifold M immersed in a 2(n+p)-dimensional indefinite complex space form $\bar{M}(c), c \neq 0$ of holomorphic sectional curvature c and index 2p and give a pinching result of the Ricci and scalar curvatures of M.

We have shown that if the Ricci curvature R is less than $\frac{c}{4}\left((n-1)+\frac{(n+1)(n+2p)}{(2n+4p-1)}\right)$ then M is totally geodesic. Moreover the scalar curvature $\rho \geq \frac{c}{4}\left(n(n-1)+\frac{(n+1)(n+2p)}{(2n+4p-1)}\right)$ and if ρ is less than $\frac{c}{4}\left(n(n-1)+\frac{(n+1)(n+2p)}{(2n+4p-1)}\right)$ them M is totally geodesic.

Keywords: Anti-invariant submanifold, Spacelike submanifold, Complex space form, totally geodesic.

INTRODUCTION

Among all submanifolds of a Kaehler manifold there are two classes; the class of anti-invariant submanifolds and the class of holomorphic submanifolds. A submanifold of a Kaehler manifold is called anti-invariant (resp. holomorphic) if each tangent space of the submanifold is mapped into the normal space (resp. itself) by the almost complex structure of the Kaehler manifold, Chen and Ogiue (1974). A Kaehler manifold of constant holomorphic sectional curvature is called a complex space form, Wali (2005).

Let $\bar{M}(c), c \neq 0$ be an indefinite complex space form of holomorphic sectional curvature c, complex dimension (n+p), $p \neq 0$ and index 2p and let M be an n-dimensional anti-invariant maximal spacelike submanifold isometrically immersed in $\bar{M}(c), c \neq 0$. We call M a spacelike submanifold if the induced metric on M from that of the ambient space is positive definite Ishihara (1988).

Let J be the almost complex structure of $\bar{M}(c), c \neq 0$. An n-dimensional Riemannian manifold M isometrically immersed in $\bar{M}(c), c \neq 0$ is called an anti-invariant submanifold of $\bar{M}(c), c \neq 0$ if each tangent space of M is mapped into the normal space by the almost complex structure J, Yano and Kon (1976).

Let h be the second fundamental form of M in $\bar{M}(c)$ and denote by S the square of the length of the second fundamental form h.

The purpose of this paper is to study an n-dimensional anti-invariant maximal spacelike submanifold M immersed in an indefinite complex space form $\bar{M}(c), c \neq 0$ and give a pinching result of the Ricci and scalar curvatures of M.

Our main result is:

Theorem: Let M be an n-dimensional compact anti-invariant maximal spacelike submanifold of $\bar{M}_p^{n+p}(c), c \neq 0$. Then if the Ricci curvature R is less than $\frac{c}{4}\left((n-1)+\frac{(n+1)(n+2p)}{(2n+4p-1)}\right)$ then M is totally geodesic. Moreover, the scalar curvature

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$\rho \geq \frac{c}{4}\left(n(n-1)+\frac{(n+1)(n+2p)}{(2n+4p-1)}\right)$ and if ρ is less than $\frac{c}{4}\left(n(n-1)+\frac{(n+1)(n+2p)}{(2n+4p-1)}\right)$ them M is totally geodesic.

LOCAL FORMULAS

We choose a local field of orthonormal frames $\{e_1,...,e_n;e_{n+1},...,e_{n+p};e_{1^*}=Je_1,...,e_{n^*}=Je_n;$

$e_{(n+1)^*}=Je_{n+1},...,e_{(n+p)^*}=Je_{n+p}\}$

in $\bar{M}_p^{n+p}(c)$ such that restricted to M, the vectors $\{e_1,...,e_n\}$ are tangent to M and the rest are normal to M. With respect to this frame field of $\bar{M}_p^{n+p}(c)$, let $\omega^1,...,\omega^n;\omega^{n+1},...,\omega^{n+p};\omega^{1^*},...,\omega^{n^*};\omega^{(n+1)^*},...,\omega^{(n+p)^*}$ be the field of dual frames.

Unless otherwise stated, we shall make use of the following convention on the ranges of indices: $1\leq A,B,C,D\leq n+p;$ $1\leq i,j,k,l,m\leq n;$ $n+1\leq a,b,c\leq n+p;$ and when a letter appears in any term as a subscript and a superscript, it is understood that this letter is summed over its range. Besides

$\varepsilon_i = g(e_i,e_i)=g(Je_i,Je_i)=1$, when $1\leq i\leq n$

$\varepsilon_a = g(e_a,e_a)=g(Je_a,Je_a)=-1$, when $n+1\leq a\leq n+p$.

Then the structure equations of $\bar{M}_p^{n+p}(c), c\neq 0$ are;

$$d\omega^A+\sum_B \varepsilon_B\omega_B^A\wedge\omega^B=0, \qquad \omega_B^A+\omega_A^B=0,$$

$$\omega_j^i=\omega_{j^*}^{i^*}, \qquad \omega_j^{i^*}=\omega_i^{j^*},$$

$$d\omega_B^A+\sum_C \varepsilon_C\omega_C^A\wedge\omega_B^C=\frac{1}{2}\sum_{CD}\varepsilon_C\varepsilon_D\bar{R}_{BCD}^A\omega^C\wedge\omega^D,$$

$$\bar{R}_{BCD}^A=\frac{c}{4}\varepsilon_C\varepsilon_D(\delta_{AC}\delta_{BD}-\delta_{AD}\delta_{BC}$$
$$+J_{AC}J_{BD}-J_{AD}J_{BC}+2J_{AB}J_{CD})$$

where $\bar{R}_{BCD}^A$ denote the components of the curvature tensor $\bar{R}$ on $\bar{M}_p^{n+p}(c), c\neq 0$.

Restricting these forms to M we have;

$$\omega^a=0,\ \ \omega_i^a=\sum_i h_{ij}^a\omega^i,\ \ h_{ij}^a=h_{ji}^a,$$

$$d\omega^i=-\sum\omega_j^i\wedge\omega^j,$$

$$\omega_j^i+\omega_i^j=0,$$

$$d\omega_j^i=-\sum\omega_k^i\wedge\omega_j^k+\frac{1}{2}\sum_{kl}R_{jkl}^i\omega^k\wedge\omega^l,$$

$$R_{jkl}^i=\bar{R}_{jkl}^i-\sum_a\left(h_{ik}^a h_{jl}^a-h_{il}^a h_{jk}^a\right),$$

$$d\omega^a=-\sum_b\omega_b^a\wedge\omega_b,$$

$$d\omega_b^a=-\sum_c\omega_c^a\wedge\omega_b^c+\frac{1}{2}R_{bij}^a\omega^i\wedge\omega^j,$$

$$R_{bij}^a=\sum_k\left(h_{ik}^a h_{kj}^b-h_{kj}^a h_{ki}^b\right) \qquad (2.1)$$

From the condition on the dimensions of M and $\bar{M}_p^{n+p}(c)$ it follows that $e_{1^*},...,e_{n^*}$ is a frame for $T^{\perp}(M)$. Noticing this, we see that

$$R_{jkl}^i=\frac{c}{4}\left(\delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk}\right)-\sum_a\left(h_{ik}^a h_{jl}^a-h_{il}^a h_{jk}^a\right) \qquad (2.2)$$

We call $H=\frac{1}{n}\sqrt{\sum_a\left(\sum_i h_{ii}^a\right)^2}$ the mean curvature of M and $S=\sum_{ija}\left(h_{ij}^a\right)^2$ the square of the length of the second fundamental form. If H is identically zero then M is said to be maximal. M is totally geodesic if h=0.

From equation (2.2) we have the Ricci tensor R_j^i given by

$$R_j^i=\sum_k R_{kjk}^i=\frac{(n-1)}{4}c\delta_{ij}+\sum_{ak}h_{ik}^a h_{kj}^a \qquad (2.3)$$

Thus the Ricci curvature R is

$$R=R_i^i=\frac{c}{4}(n-1)+S \qquad (2.4)$$

From equation (2.3) the scalar curvature ρ is given by

$$\rho=\sum_j R_j^j=\frac{n(n-1)}{4}c+S \qquad (2.5)$$

Let h_{ijk}^a denote the covariant derivative of h_{ij}^a. Then we define h_{ijk}^a by

$$\sum_k h_{ijk}^a\omega^k=dh_{ij}^a+\sum_k h_{kj}^a\omega_i^k+\sum_k h_{ik}^a\omega_j^k+\sum_b h_{ij}^b\omega_b^a \qquad (2.6)$$

and $h_{ijk}^a=h_{ikj}^a$. Taking the exterior derivative of equation (2.6) we define the second covariant derivative of h_{ij}^a by

$$\sum_l h_{ijkl}^a\omega^l=dh_{ijk}^a+\sum_l h_{ljk}^a\omega_i^l+\sum_l h_{ilk}^a\omega_j^l+\sum_l h_{ijl}^a\omega_k^l+\sum_b h_{ijk}^b\omega_b^a \qquad (2.7)$$

Using equation (2.7) we obtain the Ricci formula

$$h^a_{ijkl} - h^a_{ijlk} = \sum_m h^a_{mj} R^m_{ikl} + \sum_m h^a_{im} R^m_{jkl} + \sum_b h^b_{ij} R^a_{bkl} \quad (2.8)$$

The Laplacian Δh^a_{ij} of the second fundamental form h^a_{ij} is defined as $\Delta h^a_{ij} = \sum_k h^a_{ijkk}$.

Thus,

$$\Delta h^a_{ij} = nH_{ij} + \frac{c}{4}(n+1)\sum h^a_{ij} - \sum_{bmk} h^a_{mi} h^b_{mk} h^b_{kj} + \sum_{bmk} h^a_{mi} h^b_{mj} h^b_{kk} - \sum_{bmk} h^a_{km} h^b_{mj} h^b_{ik}$$
$$+ \sum_{bmk} h^a_{km} h^b_{mk} h^b_{ij} + \sum_{bmk} h^b_{ki} h^a_{jm} h^b_{mk} - \sum_{bmk} h^b_{ki} h^a_{mk} h^b_{mj} \quad (2.9)$$

where H_{ij} is the second covariant derivative of H.

For M maximal in $\bar{M}^{n+p}_p(c)$, equation (2.9) becomes,

$$\Delta h^a_{ij} = \frac{c}{4}(n+1)\sum h^a_{ij} - \sum_{bmk} h^a_{mi} h^b_{mk} h^b_{kj} - \sum_{bmk} h^a_{km} h^b_{mj} h^b_{ik}$$

$$+ \sum_{bmk} h^a_{km} h^b_{mk} h^b_{ij} + \sum_{bmk} h^b_{ki} h^a_{jm} h^b_{mk} - \sum_{bmk} h^b_{ki} h^a_{mk} h^b_{mj} \quad (2.10)$$

From $\frac{1}{2}\Delta \sum_{aij}\left(h^a_{ij}\right)^2 = \sum_{aijk}\left(h^a_{ijk}\right)^2 + \sum_{aij} h^a_{ij}\Delta h^a_{ij}$ we get,

$$\frac{1}{2}\Delta\sum_{aij}\left(h^a_{ij}\right)^2 = \sum_{aijk}\left(h^a_{ijk}\right)^2 + \frac{c}{4}(n+1)\sum_{aij}\left(h^a_{ij}\right)^2 - \sum_{abijkl} h^a_{ij} h^a_{kl} h^b_{lk} h^b_{ij}$$

$$+ \sum_{abijkl}\left(h^a_{li} h^b_{lj} - h^b_{li} h^a_{lj}\right)\left(h^a_{ki} h^b_{kj} - h^b_{ki} h^a_{kj}\right) \quad (2.11)$$

For each a let H_a denote the symmetric matrix $\left(h^a_{ij}\right)$. Then equation (2.11) can be written as

$$\frac{1}{2}\Delta\sum_{aij}\left(h^a_{ij}\right)^2 = \sum_{aijk}\left(h^a_{ijk}\right)^2 + \frac{c}{4}(n+1)\sum_{aij}\left(h^a_{ij}\right)^2 - \sum_{ab}\left(trH_a H_b\right)^2$$

$$+ \sum_{ab} tr\left(H_a H_b - H_b H_a\right)^2 \quad (2.12)$$

where $trH_a H_b$ denotes the trace of the matrix $H_a H_b$.

In the sequel, we need the following lemma proved by Chern *et al*. (1970).

Lemma 2.1: Let A and B be symmetric nxn-matrices. Then,

$-Tr(AB - BA)^2 \le 2TrA^2 TrB^2$ and equality holds for non-zero matrices A and B if and only if A and B can be transformed by an orthogonal matrix simultaneously into scalar multiples of $\bar{A}$ and $\bar{B}$ respectively, where

$$\bar{A} = \left(\begin{array}{cc|c} 0 & 1 & 0 \\ 1 & 0 & \\ \hline & 0 & 0 \end{array}\right), \qquad \bar{B} = \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & -1 & \\ \hline & 0 & 0 \end{array}\right).$$

Moreover, if A_1, A_2, A_3 are three symmetric nxn-matrices such that

$-Tr\left(A_a A_b - A_b A_a\right)^2 = 2TrA_a^2 TrA_b^2, \quad 1 \le a, b \le 3,$ $a \ne b,$ then at least one of the matrices A_a must be zero.

Let $S_{ab} = \sum_{abij} h^a_{ij} h^b_{ij}$. Then (n+2p)×(n+2p)-matrix $\left(S_{ab}\right)$ is symmetric and can be assumed to be diagonal for a suitable choice of $e_{n+1}, \dots, e_{n+p}$. Setting $S_a = S_{aa} = trH_a^2$ and $S = \sum_a S_a$, equation (2.12) reduces to

$$\frac{1}{2}\Delta S = \sum_{aijk}\left(h^a_{ijk}\right)^2 + \frac{c}{4}(n+1)S - \sum_{ab}\left(trH_a H_b\right)^2$$
$$+ \sum_{ab} tr\left(H_a H_b - H_b H_a\right)^2 \quad (2.13)$$

On the other hand, using Lemma 2.1 we have,

$$\frac{c}{4}(n+1)S - \sum_{ab}\left(trH_a H_b\right)^2 + \sum_{ab} tr\left(H_a H_b - H_b H_a\right)^2 \ge \frac{c}{4}(n+1)S - \sum_a S_a^2 - 2\sum_{ab} S_a S_b$$

$$= \left(\frac{(1-2n-4p)}{n+2p}S + \frac{c}{4}(n+1)\right)S + \frac{1}{(n+2p)}\sum_{a>b}\left(S_a - S_b\right)^2 \quad (2.14)$$

which, together with equation (2.13), implies that

$$\frac{1}{2}\Delta S \ge \sum_{aijk}\left(h^a_{ijk}\right)^2 + \left(\frac{(1-2n-4p)}{n+2p}S + \frac{c}{4}(n+1)\right)S \quad (2.15)$$

PROOF OF THE THEOREM

To prove our result we need the following theorem proved by Omori (1967).

Theorem 3.1: Let M be a complete Riemannian manifold with Ricci curvature bounded from below. Let f be a C^2-function which is bounded from below on M. Then for all $\varepsilon > 0$, there exists a point x in M such that, at x, $\|grad\ f\| < \varepsilon$, $\Delta f > -\varepsilon$ and $f(x) < \inf f + \varepsilon$.

Now from equation (2.4) we see that M satisfies the assumption of the theorem 3.1.

Let $f = \frac{1}{\sqrt{S+d}}$ for any positive constant d. Then f is a bounded C^∞-function on M. Thus

$$\Delta f = -\frac{1}{2} f^3 \Delta S + \frac{3}{4}\|grad\ S\|^2 f^5 \quad (3.1)$$

Let ε be any positive number. Then the above theorem implies that there is a point x in M such that, at x,

$$\frac{f^6}{4}\|grad\ S\|^2 < \varepsilon, \quad \Delta f > -\varepsilon \text{ and}$$

$$f(x) < \inf\ f + \varepsilon \tag{3.2}$$

Equations (3.1) and (3.2) give

$$\frac{\Delta S}{2(S+d)^2} < 3\varepsilon + \varepsilon(\inf\ f + \varepsilon) \tag{3.3}$$

Equation (2.15) shows that

$$\frac{1}{2}\Delta S \geq \left(\frac{(1-2n-4p)}{n+2p}S + \frac{c}{4}(n+1)\right)S \cdot$$

Substituting this into equation (3.3), we get

$$\frac{S}{(S+d)^2}\left(\frac{(1-2n-4p)}{n+2p}S + \frac{c}{4}(n+1)\right) < 3\varepsilon + \varepsilon(\inf\ f + \varepsilon)$$

When $\varepsilon \to 0$, f(x) tends to infimum and S goes to supremum. Hence, we obtain

$$\left(\left(\frac{2n+4p-1}{n+2p}\right)S - \frac{c}{4}(n+1)\right)S \geq 0 \tag{3.4}$$

Thus either S=0 or $\left(\frac{2n+4p-1}{n+2p}\right)S - \frac{c}{4}(n+1) \geq 0$ which gives $S \geq \frac{(n+1)(n+2p)}{4(2n+4p-1)}c$.

If $S < \frac{(n+1)(n+2p)}{4(2n+4p-1)}c$, then substituting this into equation (3.4) we get an impossible case. Thus S must be zero, which in turn implies that M is totally geodesic.

From equation (2.4) and $S < \frac{(n+1)(n+2p)}{4(2n+4p-1)}c$, we see that $R < \frac{c}{4}\left((n-1) + \frac{(n+1)(n+2p)}{(2n+4p-1)}\right)$ which in turn implies that M is totally geodesic.

Similarly, using equation (2.15) we have

$$\left(\frac{(2n+4p-1)}{n+2p}S - \frac{c}{4}(n+1)\right)S \geq 0 \cdot$$

From equation (2.5) and $S \geq \frac{(n+1)(n+2p)}{4(2n+4p-1)}c$ we get

$$\rho \geq \frac{c}{4}\left(n(n-1) + \frac{(n+1)(n+2p)}{(2n+4p-1)}\right).$$

If $S < \frac{(n+1)(n+2p)}{4(2n+4p-1)}c$ then using equation (2.5) gives $\rho < \frac{c}{4}\left(n(n-1) + \frac{(n+1)(n+2p)}{(2n+4p-1)}\right)$ which implies that M is totally geodesic.

CONCLUSION

In this paper we studied the geometry of an n-dimensional anti-invariant maximal spacelike submanifold M immersed in an indefinite complex space form $\bar{M}(c), c \neq 0$ and found a pinching result of the Ricci and scalar curvatures of M. In conclusion, we find that if the Ricci curvature R is less than $\frac{c}{4}\left((n-1) + \frac{(n+1)(n+2p)}{(2n+4p-1)}\right)$ then M is totally geodesic. We also found that the scalar curvature $\rho \geq \frac{c}{4}\left(n(n-1) + \frac{(n+1)(n+2p)}{(2n+4p-1)}\right)$. Moreover, if ρ is less than $\frac{c}{4}\left(n(n-1) + \frac{(n+1)(n+2p)}{(2n+4p-1)}\right)$ then M is totally geodesic.

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Revised and Accepted: Nov 3, 2010