SENRA Academic Publishers, Burnaby, British Columbia
Vol. 5, No. 1, pp. 1399-1404, Feb 2011
Online ISSN: 1920-3853; Print ISSN: 1715-9997

Canadian Journal of pure&applied sciences

BUCKLING OF PRISMATIC AND NON-PRISMATIC COLUMNS USING DIFFERENTIAL QUADRATURE METHOD

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ABSTRACT

Differential Quadrature (DQ) is a numerical method for evaluating derivatives of a sufficiently smooth function. Of the various numerical solutions, differential quadrature (DQ) method has distinguished itself because of its high accuracy, straightforward implementation and generality in a variety of problems. In this paper differential quadrature method is used to solve buckling problem of column. Critical buckling load is obtained for prismatic and non-prismatic column and various boundary conditions are applied. The obtained critical buckling load is compared with exact solution. Equally spaced and Chepeshev-Gauss-Lobatto grid points are chosen to show the effect of the grid points on the solution, also the effect of the number of grid points on the solution is studied. Direct substitution method is used to implement various boundary conditions. A treatment of clamped - free boundary conditions is shown where modified weighting coefficients formula is driven. Also the effect of the non-prismatic constant on the buckling load is studied.

Keywords: Differential quadrature, buckling, prismatic, non-prismatic.

INTRODUCTION

The basic idea of Differential Quadrature (DQ) comes from Gauss Quadrature, a useful numerical integration method. Gauss quadrature is characterized by approximating a definite integral with a weighted sum of integrand values at a group of so-called Gauss points. Extending it to finding the derivatives of various orders of a sufficiently smooth function gives rise to DQ (Bellman *et al*., 1972). In other words, the derivatives of a smooth function are approximated with weighted sums of the function values at a grid points. There are many available numerical discretization techniques, such as finite difference, finite element and finite volume which using a large number of grid points. In some practical applications, the numerical solutions of partial differential equations are required at only a few specified points in the physical domain.

The differential quadrature and its applications were rapidly developed after the late 1980. Bert and Malik (1996) presented a comprehensive review of the chronological development and the applications of the DQ method. The DQ method has been efficiently employed in a variety of problems in engineering and physical sciences up to that year.

Generalized Differential Quadrature Method

The method of DQ is developed based on the assumption that the derivatives of function with respect to a space variable of a given discrete points can be expressed as weighted linear sum of the function values at all discrete points in the domain of that variable, then the derivative of the function can be written as:

$$\frac{\partial^m f(x_i)}{\partial x^m} = \sum_{j=1}^{N} C_{ij}^{(m)} f(x_j) \qquad (1)$$

Where:
$f(x_j)$ is a function value at a grid point x_j.
$C_{ij}^{(m)}$ is a weighting coefficient for the derivative of order (m).

Once the weighting coefficients are determined, the bridge to link the derivatives in the governing differential equation and the functional values at the grid points is established.

Bellman *et al*. (1972) proposed two different approaches to compute the weighting coefficients $C_{ij}^{(m)}$ in above equation. The two approaches are based on the use of two different test functions. The Bellman's first approach assumes the test function to be $g_k(x) = x^k, k = 0,1,....,N-1$. But the second approach assumes the test function to be $g_k(x) = \frac{L_N(x)}{(x-x_k).L_N^{(1)}(x)}, k = 1,2,...N-1$ where $L_N(x)$ is the Legendre polynomial.

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In order to find simple algebraic expressions for weighting coefficients without restricting the choice of grid points, the generalized differential quadrature method was developed by Shu and Richard (1992). In generalized differential quadrature, the test functions are assumed to be Lagrange interpolated polynomial as:

$$g_k(x) = \frac{M(x)}{(x - x_k).M^{(1)}(x_k)}, k = 1, 2, ..., N \qquad (2)$$

Where:

$$M(x) = \prod_{j=1}^{N} (x - x_j)$$

and

$$M^{(1)}(x) = \frac{\partial M(x)}{\partial x} = \prod_{i=1, j=1}^{N} (x_i - x_j) \qquad (3)$$

For first order derivative of equation (1) (i.e. m=1), one can substitute from equation (2), (3) into (1) the following relationships can be established:

$$C_{ij}^{(1)} = \frac{M^{(1)}(x_i)}{(x_i - x_j).M^{(1)}(x_j)}, \text{ For } i \neq j,\ i = 1, 2, ..., N \text{ and } j = 1, 2, ..., N \qquad (4)$$

$$C_{ii}^{(1)} = \frac{M^{(2)}(x_i)}{2.M^{(1)}(x_i)}, \text{ For } i = j,\ i = 1, 2, ..., N \qquad (5)$$

Equations (4), (5) are very simple algebraic equations to compute $C_{ij}^{(1)}$ without restriction on choosing sampling grid points. However the determination of $C_{ii}^{(1)}$ requires the availability of the second order derivative of $M(x)$ which is more difficult to be obtained.

Thus the coefficient $C_{ii}^{(1)}$ can be obtained as:

$$C_{ii}^{(1)} = -\sum_{j=1, i \neq j}^{N} C_{ij}^{(1)}, \text{ For } i = 1, 2, ..., N \qquad (6)$$

Where:

$$\sum_{j=1}^{N} C_{ij}^{(1)} = 0, \text{ For } i = 1, 2, ..., N$$

Similarly the weighting coefficients for the second order (i.e. m=2) and higher order derivatives can be calculated. But from the definition of the differential operator, we have

$$\frac{\partial^{(m)} f}{\partial x^{(m)}} = \frac{\partial}{\partial x}\left(\frac{\partial^{(m-1)} f}{\partial x^{(m-1)}}\right) = \frac{\partial^{(m-1)}}{\partial x^{(m-1)}}\left(\frac{\partial f}{\partial x}\right) \qquad (7)$$

Let $[A^{(m-1)}]$, $[A^{(m)}]$ be the weighting coefficient matrices of the (m-1)th and mth order derivatives respectively. Then the application of differential quadrature approximation to equation (7) results in the following recurrence relationship:

$$\left[A^{(m)}\right] = \left[A^{(1)}\right]\left[A^{(m-1)}\right] = \left[A^{(m-1)}\right]\left[A^{(1)}\right] \qquad (8)$$

To summarize, the equation (8) and the formulations for the coefficients of first derivatives (4), (6) constitute complete formula for the determination of the weighting coefficients from the first to as high as (m-1)th order derivatives.

Choosing of grid points

In this paper, two types of grid points are selected:

Equally spaced grid points

Often convenient choice for the grid points is that of equally spaced points as shown in figure 1. These are given in X- direction as:

$$X_i = \frac{i-1}{N_x - 1}; i = 1, 2, ..., N_x \qquad (9)$$

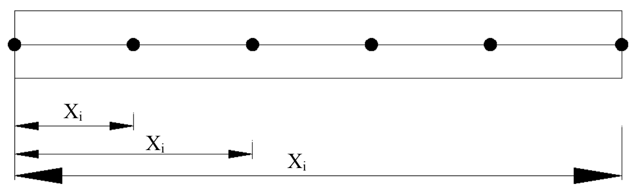

Fig. 1. Equally spaced grid points.

Chebyshev –Gauss-Lobatto grid points

Shu and Chen (1999) and Nassar *et al*. (2007) adopt Chebyshev- Gauss- Lobatto grid points as the basic mesh points to obtain an accurate solution. The coordinates of the grid points are chosen as shown in figure 2, where:

$$X_i = \frac{1}{2}\left[1 - \cos\left(\frac{i-1}{N_x - 1}\pi\right)\right]; i = 1, 2, ..., N_x \qquad (10)$$

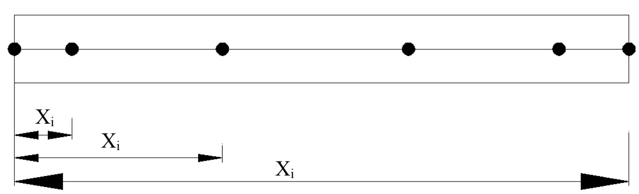

Fig. 2. Chebyshev –Gauss-Lobatto grid points.

Formulation of the problem

The general governing equation of column buckling problem, as shown in figure 3 (established by Sepahi *et al*., 2010):

$$\frac{d^2}{dx^2}\left(EI(x)\frac{d^2 w}{dx^2}\right) + P\frac{d^2 w}{dx^2} = 0 \qquad (11)$$

Where:

w is the lateral deflection of the column

P is axial load

L is the column length

E is the modulus of elasticity

I is the area moment of inertia of the column cross section

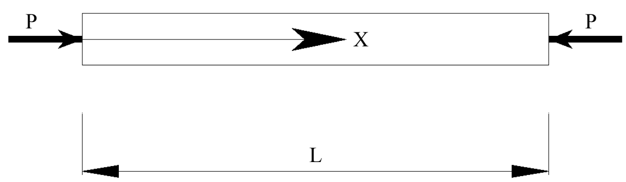

Fig. 3. Buckling of elastic column.

Non-dimensional analysis

By considering the following non-dimensional terms:

$$W = \frac{w}{L}, \quad X = \frac{x}{L}$$

One can obtain the non-dimensional governing equation:

$$\frac{d^2}{dX^2}\left(EI(X)\frac{d^2W}{dX^2}\right) + PL^2\frac{d^2W}{dX^2} = 0 \tag{12}$$

Non-Prismatic column

For non-prismatic column the governing differential equation will be:

$$I(X)\frac{d^4W}{dX^4} + 2\frac{dI(X)}{dX}\frac{d^3W}{dX^3} + \frac{d^2I(X)}{dX^2}\frac{d^2W}{dX^2} = -\frac{PL^2}{E}\frac{d^2W}{dX^2} \tag{13}$$

This differential equation can be discretized by differential quadrature as the following form:

$$I(X)\sum_{i=1,j=1}^{i=N,j=N} C_{ij}^{(4)}W_j + 2\frac{dI(X)}{dX}\sum_{i=1,j=1}^{i=N,j=N} C_{ij}^{(3)}W_j + \frac{d^2I(X)}{dX^2}\sum_{i=1,j=1}^{i=N,j=N} C_{ij}^{(2)}W_j = -\frac{PL^2}{E}\sum_{i=1,j=1}^{i=N,j=N} C_{ij}^{(2)}W_j \tag{14}$$

If the cross sectional area has the following dimensions:

$b(x) = b$ is the width of the cross section

$d(x) = d(\alpha X + 1); \alpha > 0$ is the depth at any cross section

By introducing the matrix form and Hadamard product the above equation will be in the following form:

$$[(\alpha X + 1)^3]\circ[C_{ij}^{(4)}][W_j] + [6\alpha(\alpha X + 1)^2]\circ[C_{ij}^{(3)}][W_j] + [6\alpha^2(\alpha X + 1)]\circ[C_{ij}^{(2)}][W_j] = -P^*[C_{ij}^{(2)}][W_j] \tag{15}$$

Where:

$\circ$ denote the Hadamard product.

$P^* = \frac{PL^2}{EI_0}$ is a non-dimensional axial load, I_0 is the area moment of inertia at X=0.

$[(\alpha X + 1)^3]$, $[6\alpha(\alpha X + 1)^2]$ and $[6\alpha^2(\alpha X + 1)]$ are N×N matrices whose columns are identical and each column consists of the values of the terms $[(\alpha X + 1)^3]$, $[6\alpha(\alpha X + 1)^2]$ and $[6\alpha^2(\alpha X + 1)]$ respectively at each discrete point.

From equation (15), the buckling load can be obtained by solving the above eigen-value problem together with appropriate boundary conditions.

Prismatic column

For a prismatic column (i.e.α=0) the non-dimensional governing differential equation is:

$$\frac{d^4W}{dX^4} = -\frac{PL^2}{EI}\frac{d^2W}{dX^2} \tag{16}$$

This differential equation can be discretized in the following form:

$$\sum_{i=1,j=1}^{i=N,j=N} C_{ij}^{(4)}W_j = -P^*\sum_{i=1,j=1}^{i=N,j=N} C_{ij}^{(2)}W_j \tag{17}$$

Where:

$C_{ij}^{(2)}, C_{ij}^{(4)}$ are the weighting coefficients of second and forth order.

$P^* = \frac{PL^2}{EI}$ is a non-dimensional axial load.

From equation (17), the buckling load can be obtained by solving the above eigen-value problem together with appropriate boundary conditions.

Boundary conditions

In this paper different combination of boundary conditions will be considered where:

Simply supported end (SS)

$W = 0$ and $\frac{d^2W}{dX^2} = 0$

Clamped end (C)

$W = 0$ and $\frac{dW}{dX} = 0$

Free end (F)

$\frac{d^2W}{dX^2} = 0$ and $\frac{d^3W}{dX^3} = 0$

Implementation of boundary condition

The **Direct Substitution** approach will be used in this paper, which was proposed by Shu and Du (1997).

The essence of this approach is that the Dirichlet condition is implemented at the boundary point, while the Neumman condition is descritized by the DQ method. The descritized derivative conditions at the two ends are then combined to give the W_2, $W_{(N-1)}$ in terms of $W_3, W_4, \ldots, W_{(N-2)}$. The dimension of the equation system using this approach is (N-4)×(N-4).

For any combination of the clamped and simply supported conditions, the discrete boundary conditions using the DQ method can be written as:

$$\sum_{k=1}^{N} C_{1k}^{(no)} W_k = 0 \tag{18}$$

$$\sum_{k=1}^{N} C_{Nk}^{(n1)} W_k = 0 \tag{19}$$

Where (n0), (n1) may be taken as 1 or 2. By choosing the values of n0 and n1, one can obtain the following sets of boundary conditions:

n0 = 1, n1 = 1 …clamped-----clamped
n0 = 1, n1 = 2 …clamped-----simply supported
n0 = 2, n1 = 2 …simply supported ----- simply supported

By substitution in equations (18), (19), one can couple these equations together to give W_2, $W_{(N-1)}$ as:

$$W_2 = \frac{1}{AXN} \sum_{k=3}^{N-2} AXK1 . W_k \tag{20}$$

$$W_{N-1} = \frac{1}{AXN} \sum_{k=3}^{N-2} AXKN . W_k \tag{21}$$

Where:

$$AXK1 = C_{1,k}^{(n0)} . C_{N,N-1}^{(n1)} - C_{1,N-1}^{(n0)} . C_{N,k}^{(n1)} \tag{22}$$

$$AXKN = C_{1,2}^{(n0)} . C_{N,k}^{(n1)} - C_{1,k}^{(n0)} . C_{N,2}^{(n1)} \tag{23}$$

$$AXN = C_{N,2}^{(n1)} . C_{1,N-1}^{(n0)} - C_{1,2}^{(n0)} . C_{N,N-1}^{(n1)} \tag{24}$$

Hence W_2, $W_{(N-1)}$ are expressed in terms of $W_3, W_4, \ldots, W_{(N-2)}$, and can be easily substituted into the governing discretized equations (15), (16), to be applied at (N-4) grid points, then the weighting coefficients matrices can be computed from:

$$C_1 = C_{i,k}^{(2)} + \frac{C_{i,2}^{(2)} AXK1 + C_{i,N-1}^{(2)} . AXKN}{AXN} \tag{25}$$

$$C_2 = C_{i,k}^{(3)} + \frac{C_{i,2}^{(3)} AXK1 + C_{i,N-1}^{(3)} . AXKN}{AXN} \tag{26}$$

$$C_3 = C_{i,k}^{(4)} + \frac{C_{i,2}^{(4)} AXK1 + C_{i,N-1}^{(4)} . AXKN}{AXN} \tag{27}$$

Where:
C_1 is a new weighting coefficient for second order derivative.

Table 1. Non-dimensional buckling load for prismatic column at various boundary conditions.

Boundary condition	P* (exact)	Equally spaced grid points	P*	Error%	Chebyshev Gauss Labatto grid points	P*	Error%
C---C	39.4784	N = 7 N = 11 N = 15	49.0909 39.5164 39.4784	24.3487 0.0962 0.0	N = 7 N = 11 N = 15	42.4337 39.4802 39.4784	7.485 0.0045 0.0
C---SS	20.1421	N = 7 N = 11 N = 15	19.7783 20.1865 20.1907	-1.806 0.2204 0.2412	N = 7 N = 11 N = 15	20.0720 20.1901 20.1907	-0.348 0.238 0.2412
SS---SS	9.8696	N = 7 N = 11 N = 15	10.0607 9.86970 9.86960	1.936 0.001 0.0	N = 7 N = 11 N = 15	9.9677 9.86961 9.86960	0.993 0.0001 0.0
C---F	2.4674	N = 7 N = 11 N = 15	2.4694 2.467401 2.467401	0.081 0.00004 0.00004	N = 7 N = 11 N = 15	2.4680 2.467401 2.467401	0.024 0.00004 0.00004

Table 2. Non-dimensional buckling load for non-prismatic column at various boundary conditions.

Boundary condition	Equally spaced grid points	P*	Chebyshev Gauss Lobatto grid points	P*
C---C	N = 7 N = 11 N = 15	52.3722 45.5851 45.5696	N = 7 N = 11 N = 15	48.6372 45.5706 45.5697
C---SS	N = 7 N = 11 N = 15	25.7248 23.3234 23.3077	N = 7 N = 11 N = 15	24.2536 23.3097 23.3077
SS---SS	N = 7 N = 11 N = 15	11.5779 11.3948 11.3948	N = 7 N = 11 N = 15	11.4939 11.3948 11.3948
C---F	N = 7 N = 11 N = 15	3.0053 3.0167 3.0167	N = 7 N = 11 N = 15	3.0079 3.0167 3.0167

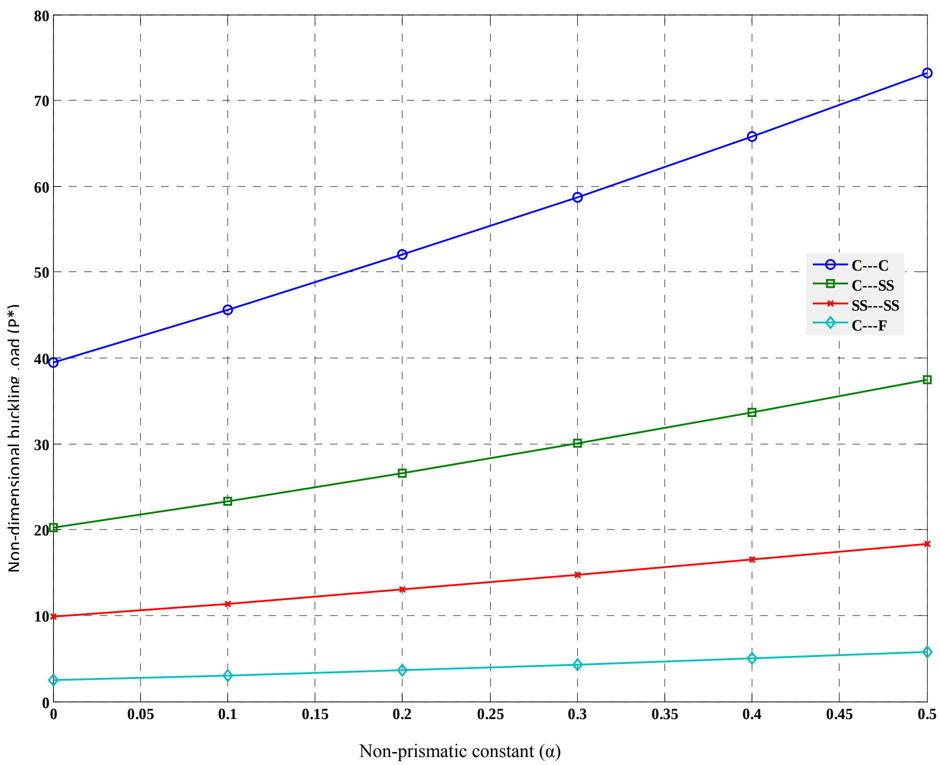

Fig. 4. Non-dimensional buckling load at different values of non-prismatic constant (α).

C_2 is a new weighting coefficient for third order derivative.
C_3 is a new weighting coefficient for fourth order derivative.

But for the case of clamped – free boundary condition, to simplify the solution, one can integrate equations (13), (16) to obtain third order differential equation and apply three boundary conditions, as, two boundaries at clamped support and moment condition at free end, then one obtain:

$$AXK1 = C_{1,k}^{(n0)} . C_{N,N}^{(n1)} - C_{1,N}^{(n0)} . C_{N,k}^{(n1)} \tag{28}$$

$$AXKN = C_{1,2}^{(n0)}.C_{N,k}^{(n1)} - C_{1,k}^{(n0)}.C_{N,2}^{(n1)} \quad (29)$$

$$AXN = C_{N,2}^{(n1)}.C_{1,N}^{(n0)} - C_{1,2}^{(n0)}.C_{N,N}^{(n1)} \quad (30)$$

Hence W_2, $W_{(N)}$ are expressed in terms of $W_3, W_4, \ldots, W_{(N-2)}$, and can be easily substituted into the governing discretized equations to be applied at (N-3) grid points, then the weighting coefficient matrices can be computed from:

$$C_1 = C_{i,k}^{(2)} + \frac{C_{i,2}^{(2)} AXK1 + C_{i,N}^{(2)}.AXKN}{AXN} \quad (31)$$

$$C_2 = C_{i,k}^{(3)} + \frac{C_{i,2}^{(3)} AXK1 + C_{i,N}^{(3)}.AXKN}{AXN} \quad (32)$$

RESULTS AND DISCUSSION

For a prismatic column

Critical non-dimensional buckling load (p^*) is obtained for a prismatic column for a different boundary conditions as shown in table 1, the effect of the types of grid points and number of grid points on the solution is studied for each case of boundary condition, also the error between the analytic and exact solution is estimated to verify DQ method.

For a non-prismatic column

Critical non-dimensional buckling load for non-prismatic column at $\alpha = 0.1$ is obtained, as shown in table 2, at different boundary conditions and different grid points type and number.

Also the values of a non-dimensional buckling load (p^*) are obtained at different non-prismatic constant (α), as shown in the figure 4.

CONCLUSION

The numerical technique of generalized differential quadrature method for the solution of differential equations has shown the great potential for being used in buckling problem because of its super accuracy, efficiency. The main advantage of this method is its inherent simplicity and the fact that easily programmable algorithmic expressions are obtained. The present method is seen to yield excellent results for the cases treated even when only a small number of grid points are used for the evaluation. Also a simple way of the treatment of clamped---free boundary condition is applied.

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Received: Sept 30, 2010; Revised: Nov 30, 2010;

Accepted: Dec 13, 2010