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ON SUBMANIFOLDS OF INDEFINITE COMPLEX SPACE FORM

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ABSTRACT

Sun (1994) showed that if M is a maximal spacelike submanifold of $\bar{M}_n^n(c)$ then either M is totally geodesic $(n \geq 2, c \geq 0)$ or $0 \leq S \leq -\frac{c}{4}n(n-1), (n \geq 2, c < 0)$. The purpose of this paper is to study the geometry of an n-dimensional compact totally real maximal spacelike submanifold M immersed in an indefinite complex space form $\bar{M}_p^{n+p}(c)$. In this manuscript we have shown that either the square of the length of second fundamental form S=0, implying M is totally geodesic for $c \geq 0, n > 1$ or $S \leq \frac{(1-n)(n+2p)}{4}c$ for $c < 0, n > 1$ and thus generalized Sun (1994) result.

Keywords: Totally real submanifold, complex space form, totally geodesic.

INTRODUCTION

Among all submanifolds of a Kaehler manifold there are two classes; the class of totally real submanifolds and the class of holomorphic submanifolds. A submanifold of a Kaehler manifold is called totally real (resp. holomorphic) if each tangent space of the submanifold is mapped into the normal space (resp. itself) by the almost complex structure of the Kaehler manifold, Chen and Ogiue (1974). A Kaehler manifold of constant holomorphic sectional curvature is called a complex space form, Wali (2005).

Let M be an n-dimensional totally real maximal spacelike submanifold isometrically immersed in a 2(n+p)-dimensional indefinite complex space form $\bar{M}_p^{n+p}(c)$ of holomorphic sectional curvature c and index 2p. We call M a spacelike submanifold if the induced metric on M from that of the ambient space is positive definite, Ishihara (1988). Let J be the almost complex structure of $\bar{M}_p^{n+p}(c)$. An n-dimensional Riemannian manifold M isometrically immersed in $\bar{M}_p^{n+p}(c)$ is called a totally real submanifold of $\bar{M}_p^{n+p}(c)$ if each tangent space of M is mapped into the normal space by the almost complex structure, Yano and Kon (1976).

Let h be the second fundamental form of M in $\bar{M}_p^{n+p}(c)$ and denote by S the square of the length of the second fundamental form h.

Sun (1994), proved that if M is a maximal spacelike submanifold of $\bar{M}_n^n(c)$ then either M is totally geodesic $(n \geq 2, c \geq 0)$ or $0 \leq S \leq -\frac{c}{4}n(n-1), (n \geq 2, c < 0)$. The purpose of this paper is to study an n-dimensional compact totally real maximal spacelike submanifold M immersed in an indefinite complex space form $\bar{M}_p^{n+p}(c)$.

Our main result is:

Theorem: Let M be an n-dimensional compact totally real maximal spacelike submanifold of $\bar{M}_p^{n+p}(c)$. Then either S=0, implying M is totally geodesic for $c \geq 0, n > 1$ or $S \leq \frac{(1-n)(n+2p)}{4}c$ for $c < 0, n > 1$.

LOCAL FORMULAS

Let $\bar{M}_p^{n+p}(c)$ be an indefinite complex space form of holomorphic sectional curvature c, dimension 2(n+p), $p \neq 0$ and index 2p. Let M be an n-dimensional totally real maximal spacelike submanifold isometrically immersed in $\bar{M}_p^{n+p}(c)$. We choose a local field of orthonormal frames

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$\left\{e_1,...,e_n;e_{n+1},...,e_{n+p};e_{1^*}=Je_1,...,e_{n^*}=Je_n;e_{(n+1)^*}=Je_{n+1},...,e_{(n+p)^*}=Je_{n+p}\right\}$

in $\bar{M}_p^{n+p}(c)$ such that restricted to M, the vectors $\left\{e_1,...,e_n\right\}$ are tangent to M and the rest are normal to M. With respect to this frame field of $\bar{M}_p^{n+p}(c)$, let $\omega^1,...,\omega^n;\omega^{n+1},...,\omega^{n+p};\omega^{1^*},...,\omega^{n^*};\omega^{(n+1)^*},...,\omega^{(n+p)^*}$ be the field of dual frames.

Unless otherwise stated, we shall make use of the following convention on the ranges of indices: $1\le A,B,C,D\le n+p;$ $1\le i,j,k,l,m\le n;$ $n+1\le \alpha,\beta,\gamma\le n+p;$ and when a letter appears in any term as a subscript or a superscript, it is understood that this letter is summed over its range. Besides

$\varepsilon_i=g\left(e_i,e_i\right)=g\left(Je_i,Je_i\right)=1$, when $1\le i\le n$

$\varepsilon_\alpha=g\left(e_\alpha,e_\alpha\right)=g\left(Je_\alpha,Je_\alpha\right)=-1$, when $n+1\le\alpha\le n+p$.

Then the structure equations of $\bar{M}_p^{n+p}(c)$ are;

$$d\omega^A+\sum\varepsilon_B\omega_B^A\wedge\omega^B=0,\ \omega_B^A+\omega_A^B=0,\ \omega_j^i=\omega_{j^*}^{i^*},\ \omega_j^{i^*}=\omega_i^{j^*},$$

$$d\omega_B^A+\sum_C\varepsilon_C\omega_C^A\wedge\omega_B^C=\frac{1}{2}\sum_{CD}\varepsilon_C\varepsilon_D\bar{R}_{ABCD}\omega^C\wedge\omega^D,$$

$$\bar{R}_{ABCD}=\frac{c}{4}\varepsilon_C\varepsilon_D\left(\delta_{AC}\delta_{BD}-\delta_{AD}\delta_{BC}+J_{AC}J_{BD}-J_{AD}J_{BC}+2J_{AB}J_{CD}\right)$$

where $\bar{R}_{ABCD}$ denote the components of the curvature tensor $\bar{R}$ on $\bar{M}_p^{n+p}(c)$.

Restricting these forms to M we have;

$$\omega^\alpha=0,\quad \omega_i^\alpha=\sum_i h_{ij}^\alpha\omega^i,\quad h_{ij}^\alpha=h_{ji}^\alpha,\ d\omega^i=-\sum\omega_j^i\wedge\omega^j,$$

$$\omega_j^i+\omega_i^j=0,$$

$$d\omega_j^i=-\sum\omega_k^i\wedge\omega_j^k+\frac{1}{2}\sum_{kl}R_{ijkl}\omega^k\wedge\omega^l,$$

$$R_{ijkl}=\bar{R}_{ijkl}-\sum_\alpha\left(h_{ik}^\alpha h_{jl}^\alpha-h_{il}^\alpha h_{jk}^\alpha\right),$$

$$d\omega^\alpha=-\sum_\beta\omega_\beta^\alpha\wedge\omega_\beta,$$

$$d\omega_\beta^\alpha=-\sum_\gamma\omega_\gamma^\alpha\wedge\omega_\beta^\gamma+\frac{1}{2}R_{\alpha\beta ij}\omega^i\wedge\omega^j,$$

$$R_{\alpha\beta ij}=\sum_k\left(h_{ik}^\alpha h_{jl}^\beta-h_{il}^\alpha h_{jk}^\beta\right)\tag{2.1}$$

From the condition on the dimensions of M and $\bar{M}_p^{n+p}(c)$ it follows that $e_{1^*},...,e_{n^*}$ is a frame for $T^\perp\left(M\right)$. Noticing this, we see that

$$R_{ijkl}=\frac{c}{4}\left(\delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk}\right)-\sum_\alpha\left(h_{ik}^\alpha h_{jl}^\alpha-h_{il}^\alpha h_{jk}^\alpha\right)\tag{2.2}$$

We call $H=\frac{1}{n}\sum_\alpha trh^\alpha$ the mean curvature of M and $S=\sum_{ij\alpha}\left(h_{ij}^\alpha\right)^2$ the square of the length of the second fundamental form. If H is identically zero then M is said to be maximal. M is totally geodesic if h=0.

From (2.2) we have the Ricci tensor R_{ij} given by

$$R_{ij}=\sum_k R_{ikjk}=\frac{\left(n-1\right)}{4}c\delta_{ij}+\sum_{\alpha k}h_{ik}^\alpha h_{kj}^\alpha\tag{2.3}$$

Thus the Ricci curvature R is

$$R=R_{ii}=\frac{c}{4}\left(n-1\right)+S\tag{2.4}$$

From (2.3) the scalar curvature is given by

$$\rho=\sum_j R_{jj}=\frac{n\left(n-1\right)}{4}c+S\tag{2.5}$$

Let h_{ijk}^α denote the covariant derivative of h_{ij}^α. Then we define h_{ijk}^α by

$$\sum_k h_{ijk}^\alpha\omega^k=dh_{ij}^\alpha+\sum_k h_{kj}^\alpha\omega_i^k+\sum_k h_{ik}^\alpha\omega_j^k+\sum_\beta h_{ij}^\beta\omega_\alpha^\beta\tag{2.6}$$

and $h_{ijk}^\alpha=h_{ikj}^\alpha$. Taking the exterior derivative of (2.6) we define the second covariant derivative of h_{ij}^α by

$$\sum_l h_{ijkl}^\alpha\omega^l=dh_{ijk}^\alpha+\sum_l h_{ljk}^\alpha\omega_i^l+\sum_l h_{ilk}^\alpha\omega_j^l+\sum_l h_{ijl}^\alpha\omega_k^l+\sum_\beta h_{ijk}^\beta\omega_\alpha^\beta\tag{2.7}$$

Using (2.7) we obtain the Ricci formula

$$h_{ijkl}^\alpha-h_{ijlk}^\alpha=\sum_m h_{mj}^\alpha R_{mikl}+\sum_m h_{im}^\alpha R_{mjkl}+\sum_\beta h_{ij}^\beta R_{\beta\alpha kl}\tag{2.8}$$

The Laplacian Δh_{ij}^α of the second fundamental form h_{ij}^α is defined as $\Delta h_{ij}^\alpha=\sum_k h_{ijkk}^\alpha$.

Therefore,

$$\Delta h_{ij}^\alpha=\frac{c}{4}\left(n-1\right)\sum h_{ij}^\alpha+\sum_{\beta mk}h_{mi}^\alpha h_{mk}^\beta h_{kj}^\beta+\sum_{\beta mk}h_{km}^\alpha h_{mk}^\beta h_{ij}^\beta\tag{2.9}$$

$$+\sum_{\beta mk}h_{ki}^\beta h_{jm}^\alpha h_{mk}^\beta-2\sum_{\beta mk}h_{ki}^\beta h_{mk}^\alpha h_{mj}^\beta$$

From $\frac{1}{2}\Delta\sum_{\alpha ij}\left(h_{ij}^{\alpha}\right)^2=\sum_{\alpha ijk}\left(h_{ijk}^{\alpha}\right)^2+\sum_{\alpha ij}h_{ij}^{\alpha}\Delta h_{ij}^{\alpha}$ we obtain,

$$\frac{1}{2}\Delta\sum_{\alpha ij}\left(h_{ij}^{\alpha}\right)^2=\sum_{\alpha ijk}\left(h_{ijk}^{\alpha}\right)^2+\frac{c}{4}(n-1)\sum_{\alpha ij}\left(h_{ij}^{\alpha}\right)^2+\sum_{\alpha\beta ijkl}h_{ij}^{\alpha}h_{kl}^{\alpha}h_{lk}^{\beta}h_{ij}^{\beta} \tag{2.10}$$

$$+\sum_{\alpha\beta ijkl}\left(h_{li}^{\alpha}h_{lj}^{\beta}-h_{li}^{\beta}h_{lj}^{\alpha}\right)\left(h_{ki}^{\alpha}h_{kj}^{\beta}-h_{ki}^{\beta}h_{kj}^{\alpha}\right)$$

PROOF OF THE THEOREM

Let M be an n-dimensional compact totally real maximal spacelike submanifold isometrically immersed in $\bar{M}_p^{n+p}(c)$. For each α let H_α denote the symmetric matrix $\left(h_{ij}^{\alpha}\right)$ and let $S_{\alpha\beta}=\sum_{ij}h_{ij}^{\alpha}h_{ij}^{\beta}$. Then the (n+2p)×(n+2p)-matrix $\left(S_{\alpha\beta}\right)$ is symmetric and can be assumed to be diagonal for a suitable choice of $e_{n+1},...,e_{n+p}$. Setting $S_\alpha=S_{\alpha\alpha}=trH_\alpha^2$ and $S=\sum_\alpha S_\alpha$, equation (2.10) can be rewritten as

$$\frac{1}{2}\Delta S=\sum_{\alpha ijk}\left(h_{ijk}^{\alpha}\right)^2+\frac{c}{4}(n-1)S+\sum_{\alpha}S_\alpha^2+\sum_{\alpha\beta}tr\left(H_\alpha H_\beta-H_\beta H_\alpha\right)^2 \tag{3.1}$$

$$=\sum_{\alpha ijk}\left(h_{ijk}^{\alpha}\right)^2+\frac{c}{4}(n-1)S+\sum_{\alpha}S_\alpha^2+\frac{1}{n+2p}S^2$$

$$+\frac{1}{n+2p}\sum_{\alpha>\beta}\left(S_\alpha-S_\beta\right)^2+\sum_{\alpha\beta}tr\left(H_\alpha H_\beta-H_\beta H_\alpha\right)^2$$

$$=\sum_{\alpha ijk}\left(h_{ijk}^{\alpha}\right)^2+\left(\frac{c}{4}(n-1)+\frac{1}{n+2p}S\right)S+\frac{1}{n+2p}\sum_{\alpha>\beta}\left(S_\alpha-S_\beta\right)^2$$

$$+\sum_{\alpha\beta}tr\left(H_\alpha H_\beta-H_\beta H_\alpha\right)^2$$

From (3.1) we see that

$$\int_M\frac{1}{2}\Delta Sdv\geq\int_M\sum_{\alpha ijk}\left(h_{ijk}^{\alpha}\right)^2dv+\int_M\left(\frac{c}{4}(n-1)+\frac{1}{n+2p}S\right)Sdv$$

where dv is the volume element of M.

By the well known theorem of Hopf (1950), $\Delta S=0$.

Therefore,

$$0\geq\int_M\sum_{\alpha ijk}\left(h_{ijk}^{\alpha}\right)^2dv+\int_M\left(\frac{c}{4}(n-1)+\frac{1}{n+2p}S\right)Sdv$$

which implies that

$$\int_M\left(\frac{c}{4}(n-1)+\frac{1}{n+2p}S\right)Sdv\leq 0 \tag{3.2}$$

Thus either S=0 implying M is totally geodesic or $S\leq\frac{(1-n)(n+2p)}{4}c$. This shows that M is totally geodesic for $c\geq 0, n>1$ or $0\leq S\leq\frac{(1-n)(n+2p)}{4}c$ for $c<0, n>1$. This proves our theorem.

CONCLUSION

In this paper we studied the geometry of an n-dimensional compact totally real maximal spacelike submanifold M immersed in an indefinite complex space form $\bar{M}_p^{n+p}(c)$ by computing the square of the length of the second fundamental form. In conclusion, we have shown that either the square of the length of second fundamental form S=0, implying M is totally geodesic for $c\geq 0, n>1$ or $S\leq\frac{(1-n)(n+2p)}{4}c$ for $c<0, n>1$. This generalizes the result by Sun (1994).

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