SENRA Academic Publishers, Burnaby, British Columbia
Vol. 3, No. 3, pp. 1001-1002, 2009
ISSN: 1715-9997

Canadian Journal of pure&applied sciences

A NEW PROOF FOR THE EULER THEOREM IN THE COMPLEX NUMBERS THEORY

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ABSTRACT

In this paper, a new proof for the Euler equation ($exp(ix) = cos\, x + i\, sin\, x$) has been presented. At first, a new and general formula has been proved from which the Euler equation has been derived.

2000 Mathematics Subject Classification: 30A99, 30B40

Keywords: Euler theorem, complex numbers, analytic function.

INTRODUCTION

Euler equation in the theory of the complex numbers is usually proved by expansion of sin(x), cos(x) and exp(x) into power series. A general proof of this equation based on direct mathematical analysis does not exist. In this paper, at first a new formula has been proved from which the Euler equation has been derived as a special result.

Analysis

Let f be an analytic function with the following characteristics

$$f(z) = u(x,y) + iv(x,y),\ f(z) \neq \pm iz_0,\ z_0 = a + ib \neq 0,\ z = x + iy,\ i = \sqrt{-1} \tag{1}$$

Since f is an analytic function [1]

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{2}$$

U and V are defined as follows

$$U(\phi,\varphi) = \frac{\phi}{\phi^2+\varphi^2}, V(\phi,\varphi) = -\frac{\varphi}{\phi^2+\varphi^2},$$

$$\phi = \phi(x,y), \varphi = \varphi(x,y) \Rightarrow$$

$$\frac{\partial U}{\partial x} = \frac{(\varphi^2-\phi^2)\frac{\partial \phi}{\partial x} - 2\phi\varphi\frac{\partial \varphi}{\partial x}}{(\phi^2+\varphi^2)^2}, \frac{\partial U}{\partial y} = \frac{(\varphi^2-\phi^2)\frac{\partial \phi}{\partial y} - 2\phi\varphi\frac{\partial \varphi}{\partial y}}{(\phi^2+\varphi^2)^2} \tag{3}$$

$$\frac{\partial V}{\partial x} = -\frac{(\phi^2-\varphi^2)\frac{\partial \varphi}{\partial x} - 2\phi\varphi\frac{\partial \phi}{\partial x}}{(\phi^2+\varphi^2)^2}, \frac{\partial V}{\partial y} = -\frac{(\phi^2-\varphi^2)\frac{\partial \varphi}{\partial y} - 2\phi\varphi\frac{\partial \phi}{\partial y}}{(\phi^2+\varphi^2)^2} \tag{3}$$

Let define g as

$$g(z) = \frac{1}{f^2(z)+z_0^2} = \frac{1}{\phi_1 + i\varphi_1} = U(\phi_1,\varphi_1)$$
$$+ iV(\phi_1,\varphi_1), \phi_1 = u^2 - v^2 + a^2 - b^2, \varphi_1 = 2(uv+ab)$$

Using Eq. 2

$$\frac{\partial \phi_1}{\partial x} = 2u\frac{\partial u}{\partial x} - 2v\frac{\partial v}{\partial x}, \frac{\partial \phi_1}{\partial y} = -2u\frac{\partial v}{\partial x} - 2v\frac{\partial u}{\partial x}, \frac{\partial \varphi_1}{\partial x} = 2u\frac{\partial v}{\partial x} + 2v\frac{\partial u}{\partial x}, \frac{\partial \varphi_1}{\partial y} = 2u\frac{\partial u}{\partial x} - 2v\frac{\partial v}{\partial x} \tag{4}$$

From Eqs. 3 and 4

$$\frac{\partial U}{\partial x} = \frac{\partial V}{\partial y} = 2\frac{(\varphi_1^2 u - \phi_1^2 u - 2\phi_1\varphi_1 v)\frac{\partial u}{\partial x} + (-\varphi_1^2 v + \phi_1^2 v - 2\phi_1\varphi_1 u)\frac{\partial v}{\partial x}}{(\phi_1^2+\varphi_1^2)^2}$$

$$\frac{\partial U}{\partial y} = -\frac{\partial V}{\partial x} = 2\frac{-(\varphi_1^2 u - \phi_1^2 u - 2\phi_1\varphi_1 v)\frac{\partial v}{\partial x} + (-\varphi_1^2 v + \phi_1^2 v - 2\phi_1\varphi_1 u)\frac{\partial u}{\partial x}}{(\phi_1^2+\varphi_1^2)^2}$$

Therefore, g is an analytic function. Let define h as follows

$$h(z) = \frac{1}{f(z)+iz_0} = \frac{1}{\phi_2 + i\varphi_2} = U(\phi_2,\varphi_2)$$

$$+ iV(\phi_2,\varphi_2), \phi_2 = u - b, \varphi_2 = v + a$$

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Using Eq. 2

$$\frac{\partial\phi_2}{\partial x}=\frac{\partial u}{\partial x},\frac{\partial\phi_2}{\partial y}=-\frac{\partial v}{\partial x},\frac{\partial\varphi_2}{\partial x}=\frac{\partial v}{\partial x},\frac{\partial\varphi_2}{\partial y}=\frac{\partial u}{\partial x} \quad (5)$$

From Eqs. 3 and 5

$$\frac{\partial U}{\partial x}=\frac{\partial V}{\partial y}=\frac{(\varphi_2^2-\phi_2^2)\frac{\partial u}{\partial x}-2\phi_2\varphi_2\frac{\partial v}{\partial x}}{(\phi_2^2+\varphi_2^2)^2},\frac{\partial U}{\partial y}$$
$$=-\frac{\partial V}{\partial x}=\frac{-(\varphi_2^2-\phi_2^2)\frac{\partial v}{\partial x}-2\phi_2\varphi_2\frac{\partial u}{\partial x}}{(\phi_2^2+\varphi_2^2)^2}$$

Therefore, h is an analytic function. Let define s as

$$s(z)=\frac{1}{f(z)-iz_0}=\frac{1}{\phi_3+i\varphi_3}$$
$$=U(\phi_3,\varphi_3)+iV(\phi_3,\varphi_3),\phi_3=u+b,\varphi_3=v-a$$

Like the procedure was used for h(z), it can be shown similarly that s(z) is also an analytic function.
Since f(z) is an analytic function, for any continuous curve C from z_0 to z (Kreyszig, 1999).

$$\int_C f(z)dz=\int_{Z_0}^{Z} f(z)dz=F(z)-F(z_0)=F(z)+c_0,F'(z)=f(z)$$
$$g(z)f'(z)=h(z)s(z)f'(z)=\frac{1}{2iz_0}(s(z)-h(z))f'(z)\Rightarrow$$
$$\int_C\frac{f'(z)dz}{f^2(z)+z_0^2}=\frac{1}{2iz_0}\int_C\left(\frac{f'(z)}{f(z)-iz_0}-\frac{f'(z)}{f(z)+iz_0}\right)dz+c_0$$
$$\Rightarrow\frac{1}{z_0}\tan^{-1}\frac{f(z)}{z_0}+c_0=\frac{1}{2iz_0}\ln\frac{f(z)-iz_0}{f(z)+iz_0}\Rightarrow\frac{f(z)-iz_0}{f(z)+iz_0}$$
$$=e^{2i\tan^{-1}\frac{f(z)}{z_0}+2ic_0z_0},\ for\ f(z)=0\Rightarrow-1=e^{2ic_0z_0}\Rightarrow$$

$$\frac{f(z)-iz_0}{f(z)+iz_0}=-e^{2i\tan^{-1}\frac{f(z)}{z_0}},\ f(z)\neq\pm iz_0 \quad (6)$$

The function f(z) can be defined as

$$f(z)=z_0\tan(p(z)/2)\Rightarrow\frac{f(z)-iz_0}{f(z)+iz_0}$$
$$=-\cos p(z)-i\sin p(z)\quad and\quad -e^{2i\tan^{-1}\frac{f(z)}{z_0}}=-e^{ip(z)}\Rightarrow$$
$$e^{ip(z)}=\cos p(z)+i\sin p(z) \quad (7)$$

For *p(z) = z,* the Laurent expansion of e^{iz} is [z]

$$e^{iz}\ [=cos(z)+i\ sin(z)]=\sum_{n=0}^{\infty}\frac{z^n}{n!}=$$
$$\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}z^{2n}+i\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}z^{2n+1}$$

This equation shows that

$$cos(z)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}z^{2n}\quad and$$
$$sin(z)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}z^{2n+1}$$

Therefore, power series expansion of *cos(z)* and *sin(z)* have been obtained without direct expansion of these functions. In contrary to the traditional procedure of proving the Euler equation, in this paper, it has been proved directly and then power series expansion of *cos(z)* and *sin(z)* has been derived from it.

REFERENCE

Kreyszig, E. 1999. Advanced Engineering Mathematics. John Wily & Sons. 669 and 717.

Received: July 13, 2009; accepted: Sept 21, 2009