SENRA Academic Publishers, Burnaby, British Columbia
Vol. 3, No. 3, pp. 959-965, 2009
ISSN: 1715-9997

Canadian Journal of pure&applied sciences

ON A CLASS OF COMPUTABLE CONVEX FUNCTIONS

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ABSTRACT

We shall show in this paper a class of computable convex functions, which have their first two solutions specified, and for which, all the polynomial solutions are uniquely determined. We shall also prove that the class of functions are convex, computable and represents a set of partial functions. Analyses indicate that it is double recursive, which can be composed from its primitive recursive functions. The class of convex functions can be shown to be reducible to Ackermann's functions with some modifications to the algorithm, which lend themselves to computability in the form of Turing machines and λ-calculus, according to Church. Least search operator or minimisation conditions can be imposed on this class of functions, such that, either no solution is returned for a certain term of the function or a term for which, the solution is zero. However, this set of computable convex functions find application in solving optimisation problems in operations research, load and demand side management in electrical power systems engineering, switching operations in computer science and electronics engineering, mathematical logic and several other application areas in industry.

Keywords: computability, optimisation, partial functions, recursive functions, turing machines.

INTRODUCTION

Information theory, electrical load and demand-side management, electronics switching techniques, optimisation, mathematical logic, nonsmooth mechanics (Moreau, 1988) and other application areas in science, engineering and industry consists in determining bounds on certain performance measures (Moon, 2000). Bounds substitute for complicated expressions that are simpler, but not exactly equal and are either larger or smaller than what they replace.

A function $f(x)$ is said to be convex over an interval (a,b) if for every $x_1, x_2 \in (a,b)$ and $0 \le \lambda \le 1$,
$$f(\lambda x_1 + (1-\lambda)x_2)) \le \lambda f(x_1) + (1-\lambda) f(x_2)$$

A function is strictly convex if equality holds only if $\lambda = 0$ or $\lambda = 1$ (Moon, 2000; Potter, 2005). One reason why we are interested in convex functions is that, it is known that over the interval of convexity, there is only one minimum. This fact can strengthen many of the results we might desire (Moon, 2000). The importance of convexity theory derives from that fact that, convex sets occur frequently in many areas of mathematics, science and engineering, and are amenable to rather elementary reasoning. In addition, the concept of convexity serves to unify a wide range of phenomena (Fink and Wood, 1996). Geometrically, every convex combination of points on the graph of the function is either above or on the graph itself. That is, in its epi-graph. This is also equivalent to saying that a function is convex iff its epi-graph is a convex set (Lebanon, 2006). Thus, we can convert a convex function on a convex set A to a convex function on R^n that is equivalent to f in some sense (Lebanon, 2006). In the same vein, a differentiable function f on a convex domain is convex iff $f(y) \ge f(x) + \nabla f(x)^T (y-x)$, that is, the graph is above the second order Taylor approximation plane. A consequence of the foregoing result is that for a convex function f, $\nabla f(x) = 0$ implies that x is a global minimum. For the second order differentiability condition: If f is twice differentiable on a convex domain A, then it is convex iff the Hessian matrix $H(x)$ is positive semi-definite for all $x \in A$.

For a convex function f and a RV X, $f(E(X) \le Ef(X)$.

The following operations preserve convexity of functions:

- A weighted combination with positive weights of convex functions is convex. If $w_i > 0$ and $f_1, ..., f_n$ are convex functions, then $\sum w_i f_i$ is convex (with a similar result for integration rather than summation). This can be seen from the second order

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condition for convexity.

- The point-wise maximum or supremum of convex functions is convex (this is a consequence of that fact that the intersection of convex epi-graphs is a convex epi-graph).
- If f is convex in (x, y) and C is a convex set, then $\inf_{y \in C} f(x, y)$ is convex in x (Beberian, 1994; Bhatia, 1997; Birge and Louveaux, 1997; Hiriart-Urruty and Lemarechal, 1996[a]; Hiriart-Urruty and Lemarechal, 1996[b]; Lebanon, 2006; Stankova-Frenkel, 2001).

RECURSIVE FUNCTIONS

Recursive functions form a class of computable functions that take their name from the process of recurrence or recursion. In general, the numerical form of recursion consists in defining the value of a function, using other values of the same function (Kasara, 2008). The Ackermann Function is a simple recursive function that produces large values from very simple inputs (Odifredi, 2005).

Proposition 1. A class of computable convex functions, which have their first two solutions specified is that for which, all the polynomial solutions are uniquely determined.

Let the first two initial solutions of the polynomial function be 1 and a.

We can define the function as $A(m,n)$:

$$m = 0,\ n = 0 \quad A(0,0) = 1$$
$$m = 0, n = 1 \quad A(0,1) = 1$$
$$m = 1, n = 0 \quad A(1,0) = 1$$
$$m = 1, n = 1 \quad A(1,1) = a$$
$$m = 1, n = 2 \quad A(1,2) = a^2$$
$$m = 2, n = 1 \quad A(2,1) = a^2$$
$$m = 2, n = 2 \quad A(2,2) = a^4$$
$$m = 2, n = 3 \quad A(2,3) = a^6$$
$$m = 3, n = 2 \quad A(3,2) = a^6$$
$$m = 3, n = 3 \quad A(3,3) = a^9$$
$$m = 3, n = 4 \quad A(3,4) = a^{12}$$
$$m = 4, n = 3 \quad A(4,3) = a^{12}$$
$$m = 4, n = 4 \quad A(4,4) = a^{16}$$
$$m = 4, n = 5 \quad A(4,5) = a^{20}$$
$$m = 5, n = 4 \quad A(5,4) = a^{20}$$
$$m = 5, n = 5 \quad A(5,5) = a^{25}$$

Therefore, $A(m,n) \equiv A(n,m) = a^{mn}$

We can begin to compute the polynomial and arrange it in an array or matrix form as follows:

n

		0	1	2	3	4	5	6	7	8	9	10			n
	0	1	1	1	1	1	1	1	1	1	1	1			1
	1	1	a	a^2	a^3	a^4	a^5	a^6	a^7	a^8	a^9	a^{10}			a^n
	2	1	a^2	a^4	a^6	a^8	a^{10}	a^{12}	a^{14}	a^{16}	a^{18}	a^{20}			a^{2n}
	3	1	a^3	a^6	a^9	a^{12}	a^{15}	a^{18}	a^{21}	a^{24}	a^{27}	a^{30}			a^{3n}
m	4	1	a^4	a^8	a^{12}	a^{16}	a^{20}	a^{24}	a^{28}	a^{32}	a^{36}	a^{40}			a^{4n}
	5	1	a^5	a^{10}	a^{15}	a^{20}	a^{25}	a^{30}	a^{35}	a^{40}	a^{45}	a^{50}		-	a^{5n}
	6	1	a^6	a^{12}	a^{18}	a^{24}	a^{30}	a^{36}	a^{42}	a^{48}	a^{54}	a^{60}		-	a^{6n}
	7	1	a^7	a^{14}	a^{21}	a^{28}	a^{35}	a^{42}	a^{49}	a^{56}	a^{63}	a^{70}	-	-	a^{7n}
	8	1	a^8	a^{16}	a^{24}	a^{32}	a^{40}	a^{48}	a^{56}	a^{64}	a^{72}	a^{80}	-	-	a^{8n}
	9	1	a^9	a^{18}	a^{27}	a^{36}	a^{45}	a^{54}	a^{63}	a^{72}	a^{81}	a^{90}	-	-	a^{9n}
	10	1	a^{10}	a^{20}	a^{30}	a^{40}	a^{50}	a^{60}	a^{70}	a^{80}	a^{90}	a^{100}	-	-	a^{10n}
	-														
	-														
	-														
	-														
	m	1	a^m	a^{2m}	a^{3m}	a^{4m}	a^{5m}	a^{6m}	a^{7m}	a^{8m}	a^{9m}	a^{10m}			a^{mn}

Both the first column and the first row can be thought of as initialisation for line 0 of the array. It appears that the function contains hardly any calculations of actual values, but mostly of their indices (Kasara, 2008).

Upon a closer look at the indices, values of each row are proportional to the next. Similarly, values of each column are also proportional to the next. The upper triangle of elements above the diagonal of the matrix is a mirror image of the lower triangle of elements just below the diagonal. The function can be said to be symmetric. The determinant of the matrix above vanishes because two rows or two columns are proportional.

The polynomial can be said to be an alternating function (Uspensky, 1948).

We can also assume the element a to be a complex quantity in order to be able to evaluate its determinant.

The determinant of the above matrix can be determined if it is a square matrix with $m = n$ (that is, $n \times n$ matrix). The value of the determinant of the above matrix is exactly as that of the matrix below:

The determinant above is equivalent to the modified Vandermonde determinant:

$$D = a^{\sum_{1}^{n} n} \begin{vmatrix} 1..a....a^2....a^3.......a^{n-1} \\ 1..a^2...a^4...a^6........a^{2(n-1)} \\ 1..a^3...a^6....a^9.......a^{3(n-1)} \\ \\ \\ 1..a^m..a^{2m}...a^{3m}......a^{m(n-1)} \end{vmatrix}$$

This Vandermonde determinant occurs often in practice as monomials of a geometric progression in each row (Wikipedia, 2008).

The simplest way to evaluate the determinant is to replace a_n by a variable a (Uspensky, 1948).

Then, the determinant becomes a polynomial $D_n(a)$ of degree $n-1$ in a. As can be seen, by expanding it by

$$\begin{array}{ccccccccccc}
a & a^2 & a^3 & a^4 & a^5 & a^6 & a^7 & a^8 & a^9 & a^{10} & a^n \\
a^2 & a^4 & a^6 & a^8 & a^{10} & a^{12} & a^{14} & a^{16} & a^{18} & a^{20} & a^{2n} \\
a^3 & a^6 & a^9 & a^{12} & a^{15} & a^{18} & a^{21} & a^{24} & a^{27} & a^{30} & a^{3n} \\
a^4 & a^8 & a^{12} & a^{16} & a^{20} & a^{24} & a^{28} & a^{32} & a^{36} & a^{40} & a^{4n} \\
a^5 & a^{10} & a^{15} & a^{20} & a^{25} & a^{30} & a^{35} & a^{40} & a^{45} & a^{50} & a^{5n} \\
a^6 & a^{12} & a^{18} & a^{24} & a^{30} & a^{36} & a^{42} & a^{48} & a^{54} & a^{60} & a^{6n} \\
a^7 & a^{14} & a^{21} & a^{28} & a^{35} & a^{42} & a^{49} & a^{56} & a^{63} & a^{70} & a^{7n} \\
a^8 & a^{16} & a^{24} & a^{32} & a^{40} & a^{48} & a^{56} & a^{64} & a^{72} & a^{80} & a^{8n} \\
a^9 & a^{18} & a^{27} & a^{36} & a^{45} & a^{54} & a^{63} & a^{72} & a^{81} & a^{90} & a^{9n} \\
a^{10} & a^{20} & a^{30} & a^{40} & a^{50} & a^{60} & a^{70} & a^{80} & a^{90} & a^{100} & a^{10n} \\
a^m & a^{2m} & a^{3m} & a^{4m} & a^{5m} & a^{6m} & a^{7m} & a^{8m} & a^{9m} & a^{10m} & a^{mn}
\end{array}$$

$$\begin{vmatrix} a....a^2...a^3..a^4.....a^5...a^6....a^7...a^8.....a^9.....a^{10}......................a^n \\ a^2..a^4...a^6...a^8....a^{10}..a^{12}..a^{14}..a^{16}.....a^{18}....a^{20}.....................a^{2n} \\ a^3..a^6...a^9...a^{12}...a^{15}..a^{18}..a^{21}...a^{24}....a^{27}....a^{30}.....................a^{3n} \\ \\ \\ \\ a^m..a^{2m}..a^{3m}..a^{4m}..a^{5m}..a^{6m}..a^{7m}..a^{8m}..a^{9m}..a^{10m}....................a^{mn} \end{vmatrix} = D$$

elements of the last row. For $a = a_1, a_2, ..., a_{n-1}$, this polynomial vanishes since $D(a_\alpha)$ for $\alpha = 1, 2, ..., n-1$ appears as determinant with two identical rows. Hence,
$D_n(a) = C(a-a_1)(a-a_2)...(a-a_{n-1})$, where C is the leading coefficient in $D_n(a)$. This coefficient is the minor

$$D_{n-1} = \begin{vmatrix} 1..a_1..a_1^2....a_1^{n-2} \\ 1...a_2....a_2^2........a_2^{n-2} \\ \\ \\ 1..a_{n-1}...a_{n-1}^2......a_{n-1}^{n-2} \end{vmatrix}$$

corresponding to a_n^{n-1}, so we have
$D_n(a_n) = D_n = D_{n-1}(a_n - a_1)(a_n - a_2)...(a_n - a_{n-1})$ *

The determinant D_{n-1} is of the same type as D_n and can be treated in the same fashion.

But,

$$D_2 = \begin{vmatrix} 1..a_1 \\ 1..a_2 \end{vmatrix} = a_2 - a_1$$

Hence, as follows from (*) for $n = 3$,
$D_3 = (a_3 - a_1)(a_3 - a_2)(a_2 - a_1)$

Further,

$D_4 = (a_4 - a_3)(a_4 - a_2)(a_4 - a_1)(a_3 - a_1)(a_3 - a_2)(a_2 - a_1)$,
etc

The general expression of Vandermonde determinant is

$$\begin{aligned} D_n = &(a_n - a_1)(a_n - a_2)....(a_n - a_{n-1}) \\ &(a_{n-1} - a_1)(a_{n-1} - a_2)....(a_{n-1} - a_{n-2}) \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &(a_3 - a_2)(a_2 - a_1) \\ &(a_2 - a_1). \end{aligned}$$

It is a rational integral function of $a_1, a_2, ..., a_n$ that mainly changes sign when two of the variables are transposed and for this reason, it is called an alternating function (Uspensky, 1948). For the exchange of two variables like a_1 and a_2 corresponds to the exchange of the first and second rows, and this causes the change of sign of the Vandermonde determinant.

For an equation with numerical coefficients, the computation of the discriminant can be reduced to the computation of a numerical determinant of the same order as the degree of the equation. If $\alpha_1, \alpha_2, ...\alpha_n$ be the roots of the equation, then the square of Vandermonde determinant becomes (Uspensky, 1948; Wong, 1997).

$$\begin{vmatrix} 1..\alpha_1..\alpha_1^2.......\alpha_1^{n-1} \\ 1..\alpha_2..\alpha_2^2.......\alpha_2^{n-1} \\ \\ \\ 1..\alpha_n..\alpha_n^2........\alpha_n^{n-1} \end{vmatrix} = (\alpha_n - \alpha_1).......(\alpha_n - \alpha_{n-1})....(\alpha_2 - \alpha_1)$$

differs from D only by the factor a_0^{2n-2}. Now multiplying Vandermonde determinant by itself, column by column, and denoting as usual by
$s_i = \alpha_1^i + \alpha_2^i + ... + \alpha_n^i$

The sum of the i th powers of roots, we have

$$\begin{vmatrix} 1..\alpha_1..\alpha_1^2......\alpha_1^{n-1} \\ 1..\alpha_2..\alpha_2^2.......\alpha_2^{n-1} \\ \\ \\ 1..\alpha_n..\alpha_n^2........\alpha_n^{n-1} \end{vmatrix} = \begin{vmatrix} s_0..s_1...........s_{n-1} \\ s_1..s_2............s_n \\ \\ \\ s_{n-1}...s_n..........s_{2n-2} \end{vmatrix},$$

So,

$$D = a_0^{2n-2} \begin{vmatrix} s_0...s_1..........s_{n-1} \\ s_1...s_2..........s_n \\ \\ \\ s_{n-1}..s_n..........s_{2n-2} \end{vmatrix}$$

The sums s_i are readily computed from Newton's formulae (Uspensky, 1948; Wong, 1997).

DOUBLE RECURSION

Primitive recursion can be used to define functions of many variables, but only by keeping all but one of them fixed. Double recursion relaxes this condition. It allows the recursion to happen on two variables instead of only one (Odifredi, 2005).

The Archimedes double recursion is reducible to primitive recursion.

$h_n(x) = a^{xn}$

But, the Ackermann's double recursion function is not reducible to primitive functions.

$$a(0,n) = n+1$$

$$a(m+1,0) = a(m,1)$$

$$a(m+1,n+1) = a(m, a(m+1,n))$$

Ackermann's function grows very fast. It can be thought of as defining a function by three arguments, $f(x, y, z)$. Thus. $f(x, y, z) = f_x(y, z)$

In this function, the argument x determines the function in the sequence $f_1, f_2, \ldots$ that needs to be used. z is the recursive parameter and y is idle. By dropping the y parameter, we obtain the Ackermann's function (Odifredi, 2005).

Minimisation (or Least Search)

By introducing minimisation or least search operator μ we are able to define a two place-function $f(x, y)$ by another function $g(x) = \mu y[f(x, y) = 0]$, where $g(x)$ returns the smallest number y, such that $f(x, y) = 0$, provided that any of the two conditions hold:

1. There actually exists at least one z such that $f(x, z) = 0$; and
2. For every $y' \leq y$, the value $f(x, y')$ exists and is positive.

If at least one of the above two conditions fails, then $g(x)$ fails to return a value and is undefined. But, from the introduction of the μ operator, we encounter a partial recursive function that might fail to be defined for some arguments (Odifredi, 2005).

Upon application to partial functions, we need to require that condition (2) above that $f(x, y')$ be defined for every $y' \leq y$. Therefore, μ is thought to try to compute in succession all values $f(x,0), f(x,1), f(x,2), \ldots$ until some m $f(x, m)$, returns 0. In such a case, an m, is returned (Odifredi, 2005).

The two cases where this procedure might fail to give a value are: (a) if no such m exists or if for some of the computations $f(x,0), f(x,1), f(x,2), \ldots$, itself fails to return a value.

We have by this means produced a class of partial functions, which can be obtained from their initial functions through composition, primitive recursion, and least search.

In addition, this class of functions turns out to belong to the class of Turing-computable functions and also to the class of $\lambda-$definable functions of Alonzo Church (Odifredi, 2005; Epstein and Carnielli, 1989).

The μ-Operator

Eliminating the bounds leads to a problem of undefined points. Consider $f(x) =$ the least y such that $y + x = 10$.

For each $x > 10$, $f(x)$ is undefined. Yet f is still computable: for $f(12)$, as an example. We can check each y in turn to see that $y + 12 \neq 10$ (Epstein and Carnielli, 1989).

We can say that there is no such y which makes $f(12)$ defined. We can step outside the system and define a better function that is defined everywhere.

Let us consider the function $h(w) =$ the least $\langle x, y, z \rangle$ such that $x, y, z > 0$ and $x^w + y^w = z^w$

For, $w = 58$, we can constructively check in turn each triple $\langle x, y, z \rangle$ to see if $x^{58} + y^{58} = z^{58}$.

We define the least search operator, also called the $\mu-$operator as:

$\mu y[f(\vec{x}, y) = 0] = z$ iff { $f(\vec{x}, z) = 0$ and

{for every $y < z$, $f(\vec{x}, y)$ is defined and > 0 (Epstein and Carnielli, 1989).

The min-Operator

We denote "$\min_y [f(\vec{x}, y) = 0]$", the smallest solution to the equation $f(\vec{x}, y) = 0$, if it exists, and is defined otherwise.

However, the min-operator is not the same as the $\mu-$operator. Let us define the primitive recursive function

$$h(x, y) = \begin{cases} x - y \ldots if \ldots y \leq x \\ 1 \ldots\ldots\ldots\ldots Otherwise \end{cases}$$

Now we define

$$g(x) = \mu y[2 \dot{-} h(x, y) = 0]$$

$g^*(x) = \min_y [2 \dot{-} h(x, y) = 0]$

Then
$g(0), g(1)$ are undefined, $g(2) = 0$
$g^*(0), g^*(1)$ are undefined, $g^*(2) = 0$

But we now define
$f(x) = \mu y[g(y).(x+1) = 0]$
$f^*(x) = \min_y [g^*(y).(x + 1) = 0]$

Then $f(x)$ is undefined for all x. But, for all x, $f^*(x) = 2$ (Epstein and Carnielli, 1989; Robinson, 1947).

The $\mu -$Operator is a Computable Operation
We choose the $\mu -$operator rather than the min-operator, because we may not be able to predict for which x, $f(x, y) = 0$, has a solution.
But with the $\mu -$operator, if $f(x, 0)$ is undefined, then $\mu y[f(x, y) = 0]$ is undefined too. This is so because we may not be able to get at trying $f(x, 1) = 0$ (Epstein and Carnielli, 1989). There is, however, a well-defined computable procedure for calculating $g(x)$, although it may not always give a result (Odifredi, 2005; Epstein and Carnielli, 1989; Robinson, 1947).

Partial Recursive Functions
Partial recursive functions are the smallest class containing the zero, successor, and projection functions and closed under composition, primitive recursion and the $\mu -$operator.

We call functions, which may (for all we know) be undefined for some inputs partial functions and may be denoted by the letters φ, ψ, ρ, etc. We may write

$\varphi(x)\downarrow$ for "φ applied to x is defined"

$\varphi(x)\not\downarrow$ for "φ applied to x is not defined".

We say that a set A or relation R is recursive if its characteristic function is recursive.
When we use the $\mu -$operator, we need to reverse the roles of 0 and 1 in the characteristic function. So, we define the representing function for a relation R to be $\overline{sg} \circ C_R$ (Odifredi, 2005; Epstein and Carnielli, 1989; Robinson, 1947).
2. It is not as restrictive as it may appear that the $\mu -$operator requires us to search for a y such that $\varphi(\vec{x}, y) = 0$. Given a relation R, we write

$\mu y_{\leq g(\vec{x})}[R(\vec{x}, y)]$

to mean $\mu y[y \leq g(\vec{x}) \wedge R(\vec{x}, y)]$ (Epstein and Carnielli, 1989).

One can, therefore, prove with rigour that a function is convex from any of the following criteria instead of guessing it from the graph.

1. Let $f(x)$ be a continuous function on an interval I. Then $f(x)$ is convex if and only if $(f(a) + f(b))/2 \geq f((a+b)/2)$ holds for all $a, b \in I$. Also, $f(x)$ is strictly convex if and only if $(f(a) + f(b))/2 > f((a+b)/2)$, whenever $a, b \in I$ and $a < b$.
2. Let $f(x)$ be a differentiable function on an interval I. Then $f(x)$ is convex if and only if $f'(x)$ is increasing on I. Also, $f(x)$ is strictly convex if and only if $f'(x)$ is strictly increasing on the interior of I.
3. Let $f(x)$ be a twice differentiable function on an interval I. Then, $f(x)$ is convex if and only if $f''(x) \geq 0$ for all $x \in I$. Also, $f(x)$ is strictly convex if and only if $f''(x) > 0$ for all x in the interior of I (Stankova-Frenkel, 2001).

Concluding Remarks
We have shown that for the class of functions considered in this study that it is both convex and computable. It has also been shown that over the interval of convexity there is only one minimum. This fact strengthens our claim of having the best solution to the application areas of interest in science, technology and industry. Also, a sum of convex functions is convex. Adding a constant or linear function to a convex function does not affect convexity. In addition, a convex function is never above its linear interpolation.

Convex functions represent a class of nonsmooth optimisation algorithms and techniques useful for getting results of very high quality in most application areas.

The importance of convexity theory derives from that fact that convex sets arise frequently in many application areas and often amenable to rather elementary reasoning. Also, the concept of convexity serves to unify a wide range of physical phenomena.

In the financial system, for example, several factors like: (i) equilibrium term structure, (ii) path dependence, and

(iii) convexity, combine to make financial engineering a rigorous discipline that comprise statistics, mathematics, economics and computer science (Kling, 2007). Convexity in the financial system is the curvature that relates value to a random variable, which will determine how the mean and the variance (volatility) of the random variable affect value. In addition, the matrix elements of A and its inverse differ only in the sign in the powers of α. It is possible to use a single computer programme to carry out both types of transformation. This is the same as carrying out a transformation of expressing the value of the coefficients in terms of the input quantities (Wong, 1997).

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Received: April 25, 2009; Accepted: Aug 27, 2009