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THE PARADOXES SHOW THE WAY: A CHANGE IN THE STRUCTURE OF LOGIC MEANS A CHANGE IN THE STRUCTURE OF RESEARCH

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ABSTRACT

In this paper, the main paradoxes are interpreted in such a way that they lead to the equivalence of a sentence and its negative $s \Leftrightarrow s^c$.

In classical logic, paradoxes define sentences that contradict themselves. These sentences support strategies for proving theorems. In fuzzy logic, paradoxes lie at the center of hyper-cubes. It can then be established that changing the framework eliminates paradoxes. Paradoxes therefore show us how to change the "teaching process" framework so as to transcend the obstacles they bring about to teaching and learning.

- The meaning of "nearby"[1] is examined in two research frameworks.
- A criterion for evaluating the "teaching process" is developed through a classification of mental activity by means of the measurement of the "fuzzy entropy" determining the internalization of concepts in every context.

Finally, the contextual approach to mathematical certainties is accepted.

Keywords: characteristic function, fuzzy logic, membership function, classical logic, entitlement system, hyper-cubes, propositional type, fuzzy set, fuzzy entropy.

INTRODUCTION

The sentences defining paradoxes cannot be predicative, which means that paradoxes are usually introduced by interrogative sentences. These sentences are defined by fuzzy sets and reveal an inconsistency. The inconsistency of the interrogative sentences has an epistemological basis and expresses neutrality in order not to influence the answer. Semantically speaking, it is thus precisely in the middle of the two meanings, either of which can be expected as an answer. Their contradictory nature is determined by the initial defining experience. This experience relates to the use of a property describing a set of objects (Anapolitanos, 1985).

When this property is not monitored, it defines a "class". If a set is to be defined by this property, the property must relate to the elements of an extant set. This gives the mathematical activity a frame of reference. The frame of reference is defined by a function determining the degree to which a given element participates in a set, which is to say it is defined by its allowing us to construct a set by recognizing its elements. Of course, the elements may well belong only partially to the set. Thus, classical logic looks to the "characteristic" function $\mu_A x = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}$ (King, 1998) as a membership function. In fuzzy logic, frameworks develop which are defined by the membership function pertaining in each case. Classical logic dominates thanks to its ability to construct logical categories which serve as a means of communication.

The Interpretation of Paradoxes

Paradoxes are sentences which reveal contradictions when they are subject to interpretation using classical logic. In a fuzzy logic context, these contradictions are eliminated, which brings into being a new mechanism for constructing mathematic concepts. It is common knowledge that, as well as using concepts, mathematics operates in the main by constructing them (Kant, 1781). The new perspective afforded by fuzzy logic introduced various new possibilities which served to maximize the possibility of constructing new concepts.

Applying modus ponens, axioms lead to sentences that do not contradict one another during the construction of mathematic concepts. The acceptance of the truth of a contradiction implies everything, allowing us to prove or reject anything in dual-value (classical) logic (1).

<u>(1) In classical logic, the truth of a contradiction $s \cap s^c$ means that any sentence P is true.</u>

Let $s \cap s^c \rightarrow s$ (α)

$s \cap s^c \rightarrow s^c$ (α)

then $[s \rightarrow s \cup P] \sim$ $s^c \rightarrow P$

s^c (α) Modus ponens

P : P anything Kosko (1987)

If $s \rightarrow P$ if this $s \rightarrow P$ doesn't stand then $s^c \rightarrow P$

s s^c

then P : P anything then P : P anything

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[1] Fuzzy linguistic labels expressed by fuzzy variables.

There is near zero tolerance for views that accept contradictions. As far as classical logic is concerned, self-refuting sentences are complex intuition that cannot construct concepts (Kant, 1781).

Although paradoxes initially serving as a mechanism for constructing mathematic concepts of some form in a working framework, they subsequently came to obstruct the construction of mathematic concepts of a different texture. The explanation of paradoxes in a fuzzy logic context creates an obstacle-free space in order to optimize mathematical activity. Knowledge is improved and disseminated better in suitably-designed teaching environments.

In a classical logic context, the claim $A \cap A^c \neq \Phi$ is unfounded. Of course, if we insist on the force of truth in our argument, we have to change the structure of logic. An argument is wrong if it is found to contain a paradox. If, however, rather than rejecting the argument, we are rejecting the dual-value and ascribing a value of the truth of sentence s instead of t (s) = 0 or t (s) = 1, a value in the closed interval [0, 1], i.e. $0 \leq t(s) \leq 1$,[2],[3] (2) then the paradoxes generalize into fuzzy sets and logic into fuzzy logic.

In fuzzy logic, the value of the sentence s, t(s) and the value of the sentence s^c, $t(s^c)$ are linked by the relationship $t(s^c) = 1 - t(s)$. Sentence s is defined by a fuzzy set A and sentence s^c by a fuzzy set A^c in which the relationship $A \cap A^c \neq \Phi$ makes sense.

Paradoxes result in the equivalence of two sentences $s \Leftrightarrow s^c$, as a conjunction of two entailments.

The most ancient paradox is the following:

"Is the Cretan philosopher (Epimenides) telling the truth or is he lying when he claims: “Every Cretan is a liar?”[4]

Let s be the sentence: "The Cretan philosopher is telling the truth"

and s^c be the sentence: "The Cretan philosopher is lying"

Interpretation—Method

If “the Cretan philosopher is telling the truth”, then, since every Cretan lies, it must be that “the Cretan philosopher is lying”, i.e., $\boxed{s \Rightarrow s^c}$ (1)

If “The Cretan philosopher is lying”, then, since every Cretan is lying, it must be that “The Cretan philosopher is telling the truth”, i.e., $\boxed{s^c \Rightarrow s}$ (2)

From (1) and (2) we end up with, $s \Leftrightarrow s^c$.

Epimenides’ paradox can be generalized for every self-referential sentence as follows:

"I tell the truth or lies when I claim that I lie”.

Let s be the sentence: "I tell the truth" s^c the sentence: "I lie"

- If I am telling the truth, the claim that I lie is true. Therefore I lie: i.e. $s \Rightarrow s^c$.
- If I am lying, then my claim to be lying is false. Therefore I am telling the truth: i.e. is, s^c=> s.

Finally, conjoining the two entailments I have, $s \Leftrightarrow s^c$.

We can see that every known paradox—and, indeed, every sentence that has still to be discovered to be internally contradictory—results in an equivalence between a sentence and its denial.

Let it be the table of truth of the logical propositional type of equivalence between two random sentences (Mytilineos, 1993).

P	Q	P <=> Q	
1	1	1	The logical operator of equivalence is true when both sentences have the same truth value, and then alone.
1	0	0	
0	1	0	
0	0	1	

- Classical logic defines an evaluation function U:Π → {0, 1}, where Π is the set of the sentences of language Γ.
- For fuzzy logic, the evaluation function is defined as: t: Π → [0, 1]

Classical logic: $s \Leftrightarrow s^c$ or $U(s) = U(s^c)$, since the values of U with S as an argument and U with s^c as an argument are, respectively, 0 or 1 and vice versa, we end up with a contradiction 0=1 or 1=0[5], (Anapolitanos, 1985).

Fuzzy logic: $s \Leftrightarrow s^c$ or $t(s) = t(s^c)$ or $t(s) = 1 - t(s)$, $[t(s^c) = 1 - t(s)]$ hold under fuzzy logic, giving us $t(s) = t(s^c) = ½$. (Klir and Bo, 1995).

There is no contradiction, rather a half-empty / half-full glass perspective.

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2 This claim is an interpretation of the law of the union of opposites, a basic law of dialectics

3 That the value of the sentence (s), t(s) lies within [0, 1] for empirical sentences is accepted by numerous theoreticians as corresponding incompletely with reality

4 This sentence is not a real paradox in terms of the classical meaning of the term.

5 Every contradiction results in a sentence of the form 0=1 or 1=0

The contradiction inherent to the self-negating sentences of the paradoxes is a statement to the effect that the mathematical object does not exist, which brings a strategy into play for constructing concepts: an indirect proof, proof by reduction ad absurdum. Applying this strategy, we make a hypothesis and then prove that the hypothesis leads to a self-contradiction (ad absurdum). We then go back to the hypothesis and disprove it[6] (Drossos and Papadopetrakis, 1985).

Only one step in this strategy is made by entailment: if (A => B) A presumes B and B is false or leads to the wrong way, then we deny A (we accuse the cause for the result].

A strategy with a similar form (which the creative subject uses to construct mathematical concepts) is where A or B presume Γ, and if Γ is false then either A is false or B is false or both are false.

But if we suppose that hypothesis Y (applying the laws of logic and mathematics) results in the sentence $s \cap s^c$, then by what authority can we conclude that the sentence is 100% wrong? Using similar powerful arguments, fuzzy logic tells us that $s \cap s^c$ are valid, except that both s and s^c are 50% true. We saw above that self-referential paradoxes correspond to the midpoint of the line connecting zero and one.

Paradoxes show that dual-value comes at a cost. We cannot always round off the description of a fact at no cost. Accuracy is exchanged for simplicity, and we pay for the exchange (Kosko, 1987).

“Paradoxes” as Hyper-Cube Centers

A single dimensional hyper-cube is the straight section of the line of real numbers from zero to one, that is [0,1].

In two dimensions, this is the unit square; in three dimensions, it is the unit solid cube and so on (Tzafestas, 1994).

Consider sentence s to be defined by a fuzzy set A. The fuzzy set A is defined as follows: Let X be a reference hyper-set X = {x_1, x_2}, $\wp$(x) the power set of X.

$\wp$(x) = {Φ, {x_1}, {x_2}, {x_1, x_2}}. A fuzzy sub-set of X, A = {(x_1, 1/3), (x_2, 3/4)}.

We define the Fit (fuzzy unit) vector of the fuzzy set fV: X → [0,1] x [0,1].

Φ = {(x_1,0), (x_2,0)}	has fit vector	(0,0)
{x_1} = {(x_1,1), (x_2,o)}	“ “ “	(1,0)
{x_2} = {(x_1,0), (x_2,1)}	“ “ “	(0,1)
{x_1,x_2} = {(x_1,1), (x_2,1)}	“ “ “	(1,1)
A = {(x_1,1/3), (x_2,3/4)}	“ “ “	(1/3, 3/4)

Geometrically illustrated as follows:

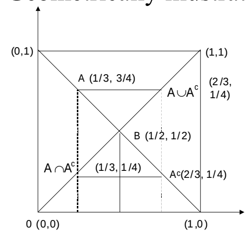

Fig. 1.

A^c = {(x_1, 2/3), (x_2, 1/4)}

$A \cap A^c$ = {(x_1, 1/3), (x_2, 1/4)}

$A \cup A^c$ = {(x_1, 2/3), (x_2, 3/4)}

The measure of entropy $E(A) = \frac{|A \cap A^c|}{|A \cup A^c|} = \frac{7}{17}$ used to gauge mental activity $0 \leq E(A) \leq 1$

For the set B = {(x_1, 1/2), (x_2, ½)} we have E(B)=1

We observe that the maximum of entropy is consumed. We conclude that the perception of the paradoxes demands the consumption of the maximum mental entropy.

The angles of the hyper-cube remain dual-valued. All other points belong to fuzzy logic Kosko (1997). At the midpoint B, the fuzzy sets B^c, $B \cap B^c$, $B \cup B^c$ are identical to B. It is the point where paradoxes are illustrated. The centers of the hyper-cubes for all dimensions are the points where the “paradoxes” reside. The geometrical illustration in the two-dimensional hyper-cube provides an oversight of the square with apexes at A, A^c, $A \cap A^c$ and $A \cup A^c$ which imbue with meaning the fuzzy sets that define the amplitude of a concept with argument sentence s.

The interpretation of paradoxes lies at the center of the hyper-cubes. This new interpretation opens up new avenues in mathematical creation as new mathematical concepts are structured diachronically. The use of fuzzy sets and of fuzzy logic in the teaching process is an everyday practice. Awareness of this process has resulted in the successful mathematization of thought using language as a tool furthering the optimization of the teaching process.

Change of context

(First) A’ Approach of the Concept of “Nearby”

We consider the sentence s: “Real numbers near five (5) included in the interval [0,10]”.

Sentence s is defined by the fuzzy set A.

$$A = \left\{ \left(x, \mu_A(x)\right) : x \in [0,10], \mu_A(x) = \frac{1}{1+(x-5)^2} \right\}$$

Sentence s^c is defined by the fuzzy set A^c.

[6] Socrates would apply this strategy to prevail over his interlocutor

$$A^c = \left\{ (x, \mu A^c(x)) : x \in [0,10], \mu A^c(x) = \frac{(x-5)^2}{1+(x-5)^2} \right\}$$

The membership function $\mu_A(x)$ is defined intuitively. The two functions can be represented thus:

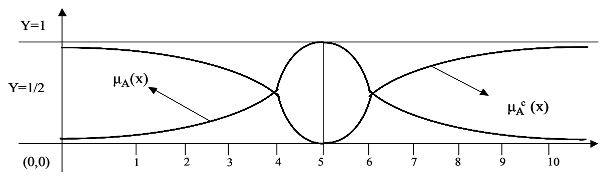

Fig. 2.

$$A \cup A^c = \left\{ (x, \mu_{A\cup A^c}(x)) : x \in [0,10], \mu_{A\cup A^c}(x) = \max\left\{ \mu_A(x), \mu_{A^c}(x) \right\} \right\}$$

and

$$A \cap A^c = \left\{ (x, \mu_{A\cap A^c}(x)) : x \in [0,10], \mu_{A\cap A^c}(x) = \min\left\{ \mu_A(x), \mu_{A^c}(x) \right\} \right\}.$$

The measure of fuzziness for fuzzy set A is defined by the price of fuzzy entropy $E(A) = \frac{|A \cap A^c|}{|A \cup A^c|}$ (Kosko, 1997).

It is calculated to be $\frac{1}{2}|A \cap A^c| = \int_0^4 \frac{1}{1+(x-5)^2} dx + \int_4^5 \frac{(x-5)^2}{1+(x-5)^2} dx =$

$$= \left[\tan^{-1}(x-5)\right]_0^4 + [x]_4^5 - \left[\tan^{-1}(x-5)\right]_4^5 \approx 0.6$$

It is also calculated to be

$$\frac{1}{2}|A \cup A^c| = \int_0^4 \frac{(x-5)^2}{1+(x-5)^2} dx + \int_4^5 \frac{1}{1+(x-5)^2} dx = 2,43$$

So $E(A) = \frac{0.6}{2.43} = 0,2,$ $\boxed{E(A) = 0.2}$

In the particular framework, which is defined by $\mu_A(x)$, the mental system spends $E(A)$ = 0.2 on interiorizing fuzzy set A.

Second Approach to the Concept of "Nearby" (vicinity)
Sentence s is defined by a fuzzy set B,

$$B = \left\{ (x, \mu_B(x)) : x \in [0,10], \mu_B(x) = \begin{cases} \frac{1}{4}x - \frac{1}{4}, & 1 \le x < 5 \\ 0 & \forall x \\ -\frac{1}{4}x + \frac{9}{4}, & 5 \le x \le 9 \end{cases} \right\}$$

Sentence s^c is defined by the fuzzy set B^c,

$$B^c = \left\{ (x, \mu_{B^c}(x)) : x \in [0,10], \mu_{B^c}(x) = \begin{cases} \frac{1}{4}x + \frac{5}{4}, & 1 \le x < 5 \\ 0 & \forall x \\ \frac{1}{4}x - \frac{5}{4}, & 5 \le x \le 9 \end{cases} \right\}$$

The two functions are illustrated thus:

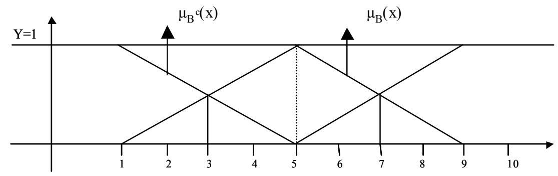

Fig. 3.

It is calculated that

$$\frac{1}{2}|B \cap B^c| = \int_1^3 \left(\frac{1}{4}x - \frac{1}{4} \right) dx + \int_3^5 \left(-\frac{1}{4}x + \frac{5}{4} \right) dx = 1$$

Also it is calculated that

$$\frac{1}{2}|B \cup B^c| = \int_1^3 \left(-\frac{1}{4}x + \frac{5}{4} \right) dx + \int_3^5 \left(\frac{1}{4}x - \frac{1}{4} \right) dx = 3$$

Thus $\boxed{E(B) = \frac{1}{3}}$

Partial Conclusions

During the mental function of interiorization (Cooper, 1739-1839) of the fuzzy sets A and B, different amounts of fuzzy entropy are expended. Fuzzy sets A and B comprise a definition for the same sentence s. The codification of the sentence is effected by means of different membership functions. The expense of mental entropy is different and scaled as follows: $E(A) < E(B)$.

The principle of maximum benefit tells us the quality of mental work is hierarchically superior when it expends less fuzzy entropy to accomplish the same mental activity.

Meaning that the teaching process must orientate itself towards developing the "membership function" in the student required for successful learning at minimum cost. This selection is often made "intuitively".

It is intuitively correct to say that the membership function's rate of transition should increase when the conviction that element x belongs to fuzzy set A is reinforced. Analytically this is expressed by the function:

$d\mu_A(x)/dx = k.\mu_A(x)(1-\mu_A(x))$

Winding this up, $\mu_A(x)=1/(1exp(a-bx))$, where the constants a, b derive from the conditions determined by the teaching process. Classical logic chooses the proper "membership function" to replace intuition, meaning the selection is made using scientific—and primarily mathematical—tools.

The development of the concept of "nearby" (vicinity) in Varied Teaching Contexts

The evaluation of whether the number (3) is near the number five (5) is clear in the first context $\mu_A(3) = \frac{1}{5} < \frac{4}{5} = \mu_{A^c}(3)$. Specifically, the number three (3) is 20% near the number five (5). Approximating, we shift to a classical logic context and resolve that the number three (3) does not belong near the number five (5).

In the second interpretational context, $\mu_B(3)=\frac{1}{2}=\mu_{B^c}(3)$. The number three (3) is 50% near the number five and is 50% not near the number five (5). Which makes it impossible to approximate in classical logic; we therefore have a "paradox". Of course, changing the context eliminates the paradox in question. But there is a paradox inherent to the structure of the new context.

Specifically, in the initial context, the reply to the question whether the number four is in the vicinity of the number five (5) is not clear if we demand an interpretation in the context of classical logic. This is the case because $\mu_A(4)=\frac{1}{5}=\mu_{A^c}(4)$. Meaning there is even equivalence for the decision posed by the question, if the number four (4) belongs to fuzzy set *A*, which defines the numbers as being in the vicinity of five (5) and of its refusal, which determines the numbers that are not in the vicinity of the number (5).

The change of context and the approximation to knowledge in a variety of contexts is ultimately a condition with a sound basis.

The use of paradoxes for backing up arguments in a research framework is common practice in mathematical creativity. It is also a tool for constructing mathematical concepts, since paradoxes function as a primitive "taboo", meaning the prohibition of a "totem". [Murder or eating a sacred animal—a totem—is prohibited, except under particular conditions with a symbolic meaning] (Freud, 1913). The paradox is a contemporary "taboo" and comprises a prohibition on a symbolic level relating to a specific "totemic" element of knowledge. The breaking of taboos, brought about by a change in context, is necessary for those seeking to structure or teach knowledge. The context changes when the membership function changes. There are paradoxes in every context which, inherent to the structure of the context, constitute axes around which knowledge is structured. Mental procedures are codified in fuzzy sets which are handled by the mental system using fuzzy logic as its method. Paradoxes lead mental procedures towards the rational.

CONCLUSIONS

Paradoxes define prohibiting values upon which rational thinking is organized.

The sentences which define paradoxes end up with the equivalence of a sentence and its negative [$s \Leftrightarrow s^c$]. In a classic logic context, the logical type defines a contradiction. It is also a criterion for rejection and a useful tool for organizing thinking. Using this tool, man is better able to construct sets. In essence, this lays the foundations for the mathematization of thinking and develops the ability to construct mathematical concepts.

- A specific paradox does not exist in every mathematical universe; it exists in a context.
- The research context is defined by a function defining an element's membership percentage in a set: the so called "membership function".
- In a classical logic context, the membership function is the typical function $\mu_A(x)=\begin{cases}1, & x\in A\\ 0, & x\notin A\end{cases}$ (King, 1998).
- There are numerous research fields in fuzzy logic. In those fields, the research context is in each case defined by the membership function. It has been discerned that paradoxes are eliminated when the research context changes in a suitable way. In the new context, other paradoxes appear which are inherent to the context.
- The change in context and the mathematic activity linked to it are a fundamental research duty for all those researching the didactics of mathematics.
- Based on the above data, the teacher has a duty to plan suitable teaching environments with a view to optimizing the teaching process. The principle of maximum benefit tells us that optimization is achieved when a concept is "interiorized" to the maximum with the minimum expense of mental activity (i.e. at the cost of the minimum "fuzzy entropy").

REFERENCES

Anapolitanos, DA. 1985. Introduction to the philosophy of mathematics, Nefeli Press, Athens.

Drossos, C. and Papadopetrakis, E. 1985. Introduction to Mathematic Logic, University of Patras, Patras.

Freud, S. 1913. Totem and Taboos, W.W. Norton & Co. Ltd., USA.

Kant, I. 1781. Kritik der reinen Vernunft in 4 Bänden, ubersetzt, hrsg. und mit Anmerkungen von Michail F. Dimitrakopoulos (1983), Athens.

Kosko, BA. 1987. Foundations of Fuzzy estimation theory, University of California, Irvine, USA.

Kosko, B. 1997. Fuzzy Logic the New Science, A. Tsoukaladakis Press, Athens.

Klir, GJ. and Bo, Y. 1995. Fuzzy sets and Fuzzy Logic, printed in the USA.

King, P. 1998. Calculating Intelligence. Eds. P. Tavlos, P. and Kostaraki, E. Athens.

Mytilineos, M. 1993. Logic, A. Stamoulis Press, Athens.

Tzafestas, Spyros. 1994. Intelligent Automatic Control, NTUA Press, Athens.