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AN INCLUSION THEOREM

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ABSTRACT

In this paper using a minimax theorem of Ricceri , we prove a result ensuring the existence of a point $x^* \in$ X such that

$$0 \in \bigcap_{\lambda \in I} F(x^*, \lambda),$$

where X is a topological space, I is a real interval and F is a multifunction from $X \times I$ to a normed space.

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The aim of this paper is to establish the following inclusion theorem:

THEOREM 1. – Let X be a topological space, Y a star-like set with respect to 0 in a normed space $(E, \|\cdot\|), I \subseteq \mathbb{R}$ an interval, $F : X \times I \to 2^Y$ a multifunction whose values are proximinal in Y. Assume that:

(a) for each $\lambda \in I$, the multifunction $F(.,\lambda)$ is open in Y and the function dist $(0, F(.,\lambda))$ is lower semicontinuous;

(b) for each $x \in X$, the function $\lambda \to dist(0, F(x,\lambda))$ is continuous and quasi-concave in I ;

(c) There exist $\rho > 0$ and $\lambda_0 \in I$ such that the set $\{x \in X : dist(0, F(x, \lambda_0)) \le \rho\}$ is compact;

(d) one has

$$0 \in \bigcap_{\lambda \in I} \bigcup_{x \in X} F(x, \lambda).$$

Under such assumption, there exists $x^* \in X$ such that

$$0 \in \bigcap_{\lambda \in I} F(x^*, \lambda).$$

Before giving the proof of theorem, let us recall some definitions, keeping the notations of Theorem 1.

The set Y is said to be star-like with respect to 0 if, for each $y \in Y$, the closed segment joining 0 and y is contained in Y. A multifunction $F : X \to 2^Y$ is said to be open in Y if, for each open set $A \subseteq X$ the set $F(A) := \bigcup_{x \in A} F(x)$ is open in Y.

A set $B \subseteq Y$ is said to be proximinal in Y if, for each $y \in Y$, there exists $z \in B$ such that

$$\|y - z\| = dist(y, B)$$

where $dist(y, B) = \inf_{v \in B} \|y - v\|$

A function $\psi : I \to \mathbb{R}$ is said to be quasi-concave if , for each $r \in \mathbb{R}$, the set $\{\lambda \in I : \psi(\lambda) > r\}$ is an interval.

Let us also recall a minimax theorem by Ricceri which is our main tool in proving Theorem 1.

Theorem 2 (Ricceri, 2001), Theorem 1 and Remark 1). – Let X,I be as in Theorem 1, and let $f : X \times I \to \mathbb{R}$ be a function satisfying the following conditions:

(i) for every $x \in X$, the function $f(x,.)$ is quasi-concave and continuous;

(ii) for every $\lambda \in I$, the function $f(.,\lambda)$ is lower semicontinuous and each of its local minima is a global minimum;

(iii) there exist $\rho > \sup_I \inf_X f$ and $\lambda_0 \in I$ such that the set $\{x \in X : f(x, \lambda_0) \le \rho\}$ is compact.

Then, one has

$$\sup_{\lambda \in I} \inf_{x \in X} f(x, \lambda) = \inf_{x \in X} \sup_{\lambda \in I} f(x, \lambda)$$

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Proof of Theorem 1. Fix $\lambda \in I$. Let us prove that the local minima of the function $dist(0, F(.,\lambda))$ are global minima. To this end, let us fix $x_0 \in X$ so that $dist(0, F(x_0,\lambda)) > 0$.

Since $F(x_0,\lambda)$ is proximinal in Y and $0 \in Y$, there exists $z_0 \in F(x_0,\lambda)$ such that

$\|z_0\| = dist(0, F(x_0,\lambda)$.

Now, let $\Omega \subseteq X$ be any open neighbourhood of x_0.

Since $F(\Omega,\lambda)$ is open in Y there is $r^* > 0$ such that

$$B(z_0,r^*)\bigcap Y \subseteq F(\Omega,\lambda),$$

where $B(z_0,r^*) = \left\{ y \in E : \|y - z_0\| \le r^* \right\}$.

If we choose $\mu \in \left]0, \min\left\{1, \frac{r^*}{\|z_0\|}\right\}\right[$, then we clearly have $(1-\mu)z_0 \in B(z_0,r^*)$ as well as $(1-\mu)z_0 \in Y$ since Y is star-like with respect to 0.Hence, $(1-\mu)z_0 \in F(\Omega,\lambda)$. From this, since $\|(1-\mu)z_0\| < \|z_0\|$, it follows that $dist(0, F(\Omega,\lambda)) < \|z_0\|$. Consequently, there exists some $\hat{x} \in \Omega$ such that

$dist(0, F(\hat{x},\lambda) < dist(0, F(x_0,\lambda))$

and so x_0 is not local minimum for the function $dist(0.F(.,\lambda))$, which clearly proves our claim. Now, observe that from (d) it directly follows

$$\sup_{\lambda\in I} \inf_{x\in X} dist(0, F(x,\lambda)) = 0 .$$

Then, taking (a),(b) and (c) into account, we see that the function $f : X \times I \to \mathbb{R}$ defined by

$f(x,\lambda) = dist(0, F(x,\lambda))$

satisfies all the assumptions of Theorem 2. Consequently, we have

$$\sup_{\lambda\in I} \inf_{x\in X} dist(0, F(x,\lambda)) = \inf_{x\in X} \sup_{\lambda\in I} dist(0, F(x,\lambda)),$$

that is

$$\inf_{x\in X} \sup_{\lambda\in I} dist(0, F(x,\lambda)) = 0 . \quad (1)$$

Finally, observe that, by (a), the function $\sup_{\lambda\in I} dist(0, F(.,\lambda))$ is lower semicontinuous, while, by (c), it has a non-empty compact sublevel set. Consequently, the infimum in (1) is attained at some $x^* \in X$. Since each set $F(x,\lambda)$ is closed in Y (since it is proximinal in it), x^* clearly satisfy the conclusion.

REFERENCES

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