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THE LINEAR SIGMA MODEL LAGRANGIAN DENSITY: FRACTIONAL FORMULATION

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ABSTRACT

In this paper, we reformulated the Linear Sigma Model Lagrangian density using fractional calculus by the left-right Riemann-Liouville fractional derivatives. We also determined fractional Euler-Lagrange and fractional Hamiltonian equations resulting from the Linear Sigma Model Lagrangian density. We found that the classical results are taken as a specific case of the fractional formulation for Euler-Lagrange and Hamiltonian equations.

Keywords: Fractional derivative, Lagrangian and Hamiltonian formulation.

INTRODUCTION

One way for generalization of ordinary integration and differentiation to random order is done by Fractional calculus (Oldham and Spanier, 1974; Podlubny, 1999; Hilfer, 2000; Miller and Ross, 1993). Nowadays, this subject is very active because it is can be used in many applications in different areas, such as biology, mathematics,, chemistry, control theory, and economics (Malinowska, 2013; Agrawal, 2002; Muslih *et al*., 2010; Laskin, 2000; Laskin, 2002; Agrawal, 2007; Jaradat, 2011). The model of linear sigma can be used for one gain some imminent into Quantum Chromodynamics in an effective low-energy theory approach (Ali, 2013). The using of linear sigma model, provides a helpful discussion on how spontaneous chiral symmetry breaking in strong interactions (Tamenaga *et al*., 2007).

Agrawal (2007) proposed generalized Euler Lagrange equation. Moreover, transversality conditions for fractional variational problems are recognized as Riesz fractional terms derivative. Baleanu and Agrawal (2006) developed a fractional Hamiltonian formulation for dynamical systems in terms of fractional Caputo derivatives. Huttner and Barnett (1992) proposed a canonical quantization approach that can be applied in the electromagnetic field. Their approach is implemented by using microscopic model in dispersive and lossy linear dielectrics. In this model, the medium is considered as a set of interacting matter fields. Gray (1982) presented a gauge invariant approach to quantize the electromagnetic Lagrangian density. The proposed approach is done by interpreting the Fourier coefficients of the magnetic induction field B and the coefficients of the electric field E as generalized coordinates and conjugate momenta, respectively. Furthermore, Rabei *et al*. (2007) obtained the Hamiltonian equations of motion for continuous and discrete systems by using the formulation of Euler-Lagrange equations from variational problems. Jaradat *et al*. (2010) reformulated the free electromagnetic Lagrangian density using the radiation (coulomb) gauge, and Lorentz gauge. They also obtained fractional Euler-Lagrangian equations resulting from these Lagrangian densities, and then found fractional Hamiltonian density in general form. Al–Oqali (2015) obtained fractional Euler-Lagrange equations and fractional Hamilton's equations for Higgs field using the Left-Right Riemann-Liouville fractional derivative .

This work aims to reformulate the Linear Sigma Model Lagrangian density in fractional form in terms of the Riemann-Liouville fractional derivative, and to obtain equations of motion. We also compare equations of motion obtained with the Hamilton's equations of motion in fractional form .

The remaining of this paper is organized as follows; the following sections present some basic definition of the Riemann-Liouville fractional derivative and the fractional Linear Sigma Model Lagrangian density. The next section presents the fractional Euler-Lagrange equations. After that, the fractional Hamiltonian is constructed for the system and the Hamiltonian equations of motion are obtained. Finally, the last section dedicated to our conclusions.

BASIC DEFINITIONS

In this part of the paper, we briefly present some fundamental definitions used in this work. The left and right Riemann- Liouville fractional derivatives are defined as follows (Diab *et al*., 2013):
The left Riemann- Liouville fractional derivative

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$$ {}_aD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t (t-\tau)^{n-\alpha-1} f(\tau)\,d\tau \quad (1) $$

The right Riemann- Liouville fractional derivative

$$ {}_aD_b^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \left(-\frac{d}{dt}\right)^n \int_a^t (t-\tau)^{n-\alpha-1} f(\tau)\,d\tau \quad (2) $$

Here α is the order of derivative such that n-1< α ≤ n and Γ represents the gamma function. If α is an integer, these derivatives become the usual derivatives.

$$ {}_aD_t^\alpha f(t) = \left(\frac{d}{dt}\right)^\alpha f(t) \quad (3) $$

$$ {}_aD_b^\alpha f(t) = \left(-\frac{d}{dt}\right)^\alpha f(t) \quad (4) $$

Fractional Linear Sigma Model Lagrangian Density

The Lagrangian density of the linear sigma model has the form (Gross, 1993)

$$ \mathcal{L} = \frac{1}{2}\bar{\psi} i\gamma^\nu \frac{\partial\psi}{\partial x^\nu} - \frac{1}{2}\frac{\partial\bar{\psi}}{\partial x^\nu}\gamma^\nu i\psi - g_\pi \psi\phi\bar{\psi} + \frac{1}{4} tr(\partial_\nu\phi\partial^\nu\phi^\dagger) - \frac{1}{2}m^2|\phi|^2 + V(|\phi|^2) \quad (5) $$

Where

$$ \begin{cases} V(|\phi|^2) = -\frac{1}{4}\lambda^2\phi^2\phi^{\dagger 2} \\ \frac{1}{2} tr(\partial_\nu\phi\partial^\nu\phi^\dagger) = \partial_\nu\phi\partial^\nu\phi^\dagger \end{cases} \quad (6) $$

This Lagrangian contains three parameters: one fermion-meson coupling constant g_π, a meson mass m, and a ϕ^4 type meson-meson interaction strength λ^2 .

Now the Lagrangian density becomes:

$$ \mathcal{L} = \frac{1}{2}\bar{\psi} i\gamma^\nu \frac{\partial\psi}{\partial x^\nu} - \frac{1}{2}\frac{\partial\bar{\psi}}{\partial x^\nu}\gamma^\nu i\psi - g_\pi \psi\phi\bar{\psi} + \frac{1}{2}\partial_\nu\phi\partial^\nu\phi^\dagger - \frac{1}{2}m^2\phi\phi^\dagger + -\frac{1}{4}\lambda^2\phi^2\phi^{\dagger 2} \quad (7) $$

Use the definition of left Riemann – Liouville fractional derivative, the fractional Linear Sigma Model Lagrangian Density takes the form

$$ \mathcal{L} = \frac{1}{2}\bar{\psi} i\gamma^\nu \,{}_aD_{x_\nu}^\mu \psi - \frac{1}{2}\psi i\gamma^\nu \,{}_aD_{x_\nu}^\mu \bar{\psi} - g_\pi\psi\phi\bar{\psi} + \frac{1}{2}\,{}_aD_{x_\nu}^\mu\phi \,{}_aD_{x^\nu}^\mu\phi^\dagger - \frac{1}{2}m^2\phi\phi^\dagger - \frac{1}{4}\lambda^2\phi^2\phi^{\dagger 2} \quad (8) $$

Fractional Euler – Lagrange Equations for Linear Sigma Model

Consider the action function of the form

$$ \mathcal{J} = \int \mathcal{L}\left(\psi, \phi, \phi^\dagger, \bar{\psi}, {}_aD_{x_\nu}^\alpha \psi, {}_aD_{x_\nu}^\alpha \phi, {}_aD_{x_\nu}^\alpha \phi^\dagger, {}_aD_{x_\nu}^\alpha \bar{\psi}, x\right) \quad (9) $$

Calculating $\partial\mathcal{J}$ form above equation we obtain,

$$ \partial\mathcal{J} = \int d^4x \left\{ \begin{array}{c} \frac{\partial\mathcal{L}}{\partial\psi}\delta\psi + \frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial\phi^\dagger}\delta\phi^\dagger + \frac{\partial\mathcal{L}}{\partial\bar{\psi}}\delta\bar{\psi} \\ + \frac{\partial\mathcal{L}}{\partial\,{}_aD_{x_\nu}^\alpha\psi}\delta\,{}_aD_{x_\nu}^\alpha\psi + \frac{\partial\mathcal{L}}{\partial\,{}_aD_{x_\nu}^\alpha\phi}\delta\,{}_aD_{x_\nu}^\alpha\phi + \\ \frac{\partial\mathcal{L}}{\partial\,{}_aD_{x_\nu}^\alpha\phi^\dagger}\delta\,{}_aD_{x_\nu}^\alpha\phi^\dagger + \frac{\partial\mathcal{L}}{\partial\,{}_aD_{x_\nu}^\alpha\bar{\psi}}\delta\,{}_aD_{x_\nu}^\alpha\bar{\psi} \end{array} \right\} = 0 \quad (10) $$

Using

$\delta\,{}_aD_{x_\nu}^\alpha\psi = {}_aD_{x_\nu}^\alpha\delta\psi$, $\delta\,{}_aD_{x_\nu}^\alpha\phi = {}_aD_{x_\nu}^\alpha\delta\phi$,

$\delta\,{}_aD_{x_\nu}^\alpha\phi^\dagger = {}_aD_{x_\nu}^\alpha\delta\phi^\dagger$

and $\delta\,{}_aD_{x_\nu}^\alpha\bar{\psi} = {}_aD_{x_\nu}^\alpha\delta\bar{\psi}$

We obtain,

$$ \partial\mathcal{J} = \int d^4x \left\{ \begin{array}{c} \frac{\partial\mathcal{L}}{\partial\psi}\delta\psi + \frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial\phi^\dagger}\delta\phi^\dagger + \frac{\partial\mathcal{L}}{\partial\bar{\psi}}\delta\bar{\psi} \\ + \underbrace{\frac{\partial\mathcal{L}}{\partial\,{}_aD_{x_\nu}^\alpha\psi}\,{}_aD_{x_\nu}^\alpha\delta\psi}_{fifth} + \underbrace{\frac{\partial\mathcal{L}}{\partial\,{}_aD_{x_\nu}^\alpha\phi}\,{}_aD_{x_\nu}^\alpha\delta\phi}_{sixth} + \\ \underbrace{\frac{\partial\mathcal{L}}{\partial\,{}_aD_{x_\nu}^\alpha\phi^\dagger}\,{}_aD_{x_\nu}^\alpha\delta\phi^\dagger}_{seventh} + \underbrace{\frac{\partial\mathcal{L}}{\partial\,{}_aD_{x_\nu}^\alpha\bar{\psi}}\,{}_aD_{x_\nu}^\alpha\delta\bar{\psi}}_{eighth} \end{array} \right\} = 0 \quad (11) $$

Integrating the (fifth, sixth, seventh and eighth) terms by parts give

$$ \partial\mathcal{J} = \int d^4x \left\{ \begin{array}{c} \frac{\partial\mathcal{L}}{\partial\psi}\delta\psi + \frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial\phi^\dagger}\delta\phi^\dagger + \frac{\partial\mathcal{L}}{\partial\bar{\psi}}\delta\bar{\psi} \\ - {}_aD_{x_\nu}^\alpha\left(\frac{\partial\mathcal{L}}{\partial\,{}_aD_{x_\nu}^\alpha\psi}\right)\delta\psi - {}_aD_{x_\nu}^\alpha\left(\frac{\partial\mathcal{L}}{\partial\,{}_aD_{x_\nu}^\alpha\phi}\right)\delta\phi - \\ {}_aD_{x_\nu}^\alpha\left(\frac{\partial\mathcal{L}}{\partial\,{}_aD_{x_\nu}^\alpha\phi^\dagger}\right)\delta\phi^\dagger - {}_aD_{x_\nu}^\alpha\left(\frac{\partial\mathcal{L}}{\partial\,{}_aD_{x_\nu}^\alpha\bar{\psi}}\right)\delta\bar{\psi} \end{array} \right\} = 0 \quad (12) $$

This lead to Euler – Lagrange equations.

$$ \frac{\partial\mathcal{L}}{\partial\psi} - {}_aD_t^\alpha\left(\frac{\partial\mathcal{L}}{\partial\,{}_aD_t^\alpha\psi}\right) - {}_aD_{x_i}^\alpha\left(\frac{\partial\mathcal{L}}{\partial\,{}_aD_{x_i}^\alpha\psi}\right) = 0 \quad (13) $$

$$ \frac{\partial\mathcal{L}}{\partial\phi} - {}_aD_t^\alpha\left(\frac{\partial\mathcal{L}}{\partial\,{}_aD_t^\alpha\phi}\right) - {}_aD_{x_i}^\alpha\left(\frac{\partial\mathcal{L}}{\partial\,{}_aD_{x_i}^\alpha\phi}\right) = 0 \quad (14) $$

$$ \frac{\partial\mathcal{L}}{\partial\phi^\dagger} - {}_aD_t^\alpha\left(\frac{\partial\mathcal{L}}{\partial\,{}_aD_t^\alpha\phi^\dagger}\right) - {}_aD_{x_i}^\alpha\left(\frac{\partial\mathcal{L}}{\partial\,{}_aD_{x_i}^\alpha\phi^\dagger}\right) = 0 \quad (15) $$

$$ \frac{\partial\mathcal{L}}{\partial\bar{\psi}} - {}_aD_t^\alpha\left(\frac{\partial\mathcal{L}}{\partial\,{}_aD_t^\alpha\bar{\psi}}\right) - {}_aD_{x_i}^\alpha\left(\frac{\partial\mathcal{L}}{\partial\,{}_aD_{x_i}^\alpha\bar{\psi}}\right) = 0 \quad (16) $$

So the equations of motion become

$$ \left(g_\pi\phi + i\gamma^0\,{}_aD_t^\alpha + i\gamma^i\,{}_aD_{x_i}^\alpha\right)\bar{\psi} = 0 \quad (17) $$

$$ \left({}_aD_t^\alpha\,{}_aD_t^\alpha + {}_aD_{x_i}^\alpha\,{}_aD_{x_i}^\alpha + m^2 + \lambda^2\phi\ \phi^\dagger\right)\phi^\dagger + 2g_\pi\psi\bar{\psi} = 0 \quad (18) $$

$$ \left({}_aD_t^\alpha\,{}_aD_t^\alpha + {}_aD_{x_i}^\alpha\,{}_aD_{x_i}^\alpha + m^2 + \lambda^2\phi\ \phi^\dagger\right)\phi = 0 \quad (19) $$

$$ \left(i\gamma^0\,{}_aD_t^\alpha + i\gamma^i\,{}_aD_{x_i}^\alpha - g_\pi\phi\right)\psi = 0 \quad (20) $$

For $\alpha = 1$ we have ${}_aD^{\alpha}_{x_v} = \partial_v$ and the equations (17, 18, 19, and 20) reduces to standard Euler – Lagrange equation.

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_v\left(\frac{\partial \mathcal{L}}{\partial(\partial_v \phi)}\right) = 0$$
$$\frac{\partial \mathcal{L}}{\partial \psi} - \partial_v\left(\frac{\partial \mathcal{L}}{\partial(\partial_v \psi)}\right) = 0$$
$$\frac{\partial \mathcal{L}}{\partial \phi^{\dagger}} - \partial_v\left(\frac{\partial \mathcal{L}}{\partial(\partial_v \phi^{\dagger})}\right) = 0$$
$$\frac{\partial \mathcal{L}}{\partial \bar{\psi}} - \partial_v\left(\frac{\partial \mathcal{L}}{\partial(\partial_v \bar{\psi})}\right) = 0$$

So the equations of motion become,

$$(\square + m^2 + \lambda^2 \phi \quad \phi^{\dagger})\phi^{\dagger} + 2g_{\pi}\psi\bar{\psi} = 0$$
$$(\square + m^2 + \lambda^2 \phi \quad \phi^{\dagger})\phi \quad = 0$$
$$(g_{\pi}\phi + i\gamma^{v}\partial_v)\bar{\psi} = 0$$
$$(i\gamma^{v}\partial_v - g_{\pi}\phi)\psi = 0$$

Fractional Hamiltonian Formulation

To construct the fractional Hamiltonian equation within Riemann –Liouville fractional derivative from fractional Linear Sigma Model Lagrangian density, we consider the Lagrangian depending on fractional time derivatives of coordinates in the form

$$\mathcal{L} = \mathcal{L}\left(\begin{matrix}\psi, {}_aD^{\alpha}_t\psi, {}_aD^{\alpha}_{x_j}\psi, \phi, {}_aD^{\alpha}_t\phi, {}_aD^{\alpha}_{x_j}\phi, \\ \phi^{\dagger}, {}_aD^{\alpha}_t\phi^{\dagger}, {}_aD^{\alpha}_{x_j}\phi^{\dagger}, \bar{\psi}, {}_aD^{\alpha}_t\bar{\psi}, {}_aD^{\alpha}_{x_j}\bar{\psi}\end{matrix}\right) \quad (21)$$

The Hamiltonian depending on the fractional time derivatives reads as,

$$\mathcal{H} = \pi_{\psi}\, {}_aD^{\alpha}_t\psi + \pi_{\phi}\, {}_aD^{\alpha}_t\phi + \pi_{\phi^{\dagger}}\, {}_aD^{\alpha}_t\phi^{\dagger} + \pi_{\bar{\psi}}\, {}_aD^{\alpha}_t\bar{\psi} - \mathcal{L} \quad (22)$$

Where

$$\mathcal{H} = \mathcal{H}\left(\begin{matrix}\psi, \pi_{\psi}, {}_aD^{\alpha}_{x_j}\psi, \phi, \pi_{\phi}, {}_aD^{\alpha}_{x_j}\phi \\ \phi^{\dagger}, \pi_{\phi^{\dagger}}, {}_aD^{\alpha}_{x_j}\phi^{+}, \bar{\psi}, \pi_{\bar{\psi}}, {}_aD^{\alpha}_{x_j}\bar{\psi}\end{matrix}\right) \quad (23)$$

Thus, the total differential of Hamiltonian function reads as

$$d\mathcal{H} = \left\{\begin{matrix}\frac{\partial \mathcal{H}}{\partial \psi}d\psi + \frac{\partial \mathcal{H}}{\partial \pi_{\psi}}d\pi_{\psi} + \frac{\partial \mathcal{H}}{\partial\left({}_aD^{\alpha}_{x_j}\psi\right)}d\left({}_aD^{\alpha}_{x_j}\psi\right) + \\ \frac{\partial \mathcal{H}}{\partial \phi}d\phi + \frac{\partial \mathcal{H}}{\partial \pi}d\pi + \frac{\partial \mathcal{H}}{\partial\left({}_aD^{\alpha}_{x_j}\phi\right)}d\left({}_aD^{\alpha}_{x_j}\phi\right) + \\ \frac{\partial \mathcal{H}}{\partial \phi^{\dagger}}d\phi^{\dagger} + \frac{\partial \mathcal{H}}{\partial\left({}_aD^{\alpha}_{x_j}\phi^{\dagger}\right)}d\left({}_aD^{\alpha}_{x_j}\phi^{\dagger}\right) + \frac{\partial \mathcal{H}}{\partial \pi^{\dagger}}d\pi^{\dagger} + \\ \frac{\partial \mathcal{H}}{\partial \bar{\psi}}d\bar{\psi} + \frac{\partial \mathcal{H}}{\partial \pi_{\bar{\psi}}}d\pi_{\bar{\psi}} + \frac{\partial \mathcal{H}}{\partial\left({}_aD^{\alpha}_{x_j}\bar{\psi}\right)}d\left({}_aD^{\alpha}_{x_j}\bar{\psi}\right)\end{matrix}\right\} \quad (24)$$

and

$$d\mathcal{H} = \begin{cases}\pi_{\psi}d\left({}_aD^{\alpha}_t\psi\right) + {}_aD^{\alpha}_t\psi\, d\pi_{\psi} + \pi_{\phi}d\left({}_aD^{\alpha}_t\phi\right) + {}_aD^{\alpha}_t\phi\, d\pi_{\phi} \\ +\pi_{\phi^{\dagger}}d\left({}_aD^{\alpha}_t\phi^{\dagger}\right) + {}_aD^{\alpha}_t\phi^{\dagger}d\pi_{\phi^{\dagger}} - \frac{\partial \mathcal{L}}{\partial \psi}d\psi - \pi_{\psi}d\left({}_aD^{\alpha}_t\psi\right) + \\ {}_aD^{\alpha}_t\psi\, d\pi_{\psi} - \frac{\partial \mathcal{L}}{\partial\left({}_aD^{\alpha}_t\psi\right)}d\left({}_aD^{\alpha}_t\psi\right) - \frac{\partial \mathcal{L}}{\partial\left({}_aD^{\alpha}_{x_j}\psi\right)}d\left({}_aD^{\alpha}_{x_j}\psi\right) - \\ \frac{\partial \mathcal{L}}{\partial \phi}d\phi - \frac{\partial \mathcal{L}}{\partial\left({}_aD^{\alpha}_t\phi\right)}d\left({}_aD^{\alpha}_t\phi\right) - \frac{\partial \mathcal{L}}{\partial\left({}_aD^{\alpha}_{x_j}\phi\right)}d\left({}_aD^{\alpha}_{x_j}\phi\right) - \\ \frac{\partial \mathcal{L}}{\partial \phi^{\dagger}}d\phi^{\dagger} - \frac{\partial \mathcal{L}}{\partial\left({}_aD^{\alpha}_t\phi^{\dagger}\right)}d\left({}_aD^{\alpha}_t\phi^{\dagger}\right) - \\ \frac{\partial \mathcal{L}}{\partial\left({}_aD^{\alpha}_{x_j}\phi^{+}\right)}d\left({}_aD^{\alpha}_{x_j}\phi^{+}\right) - \frac{\partial \mathcal{L}}{\partial \bar{\psi}}d\bar{\psi} - \frac{\partial \mathcal{L}}{\partial\left({}_aD^{\alpha}_t\bar{\psi}\right)}d\left({}_aD^{\alpha}_t\bar{\psi}\right) - \\ \frac{\partial \mathcal{L}}{\partial\left({}_aD^{\alpha}_{x_j}\bar{\psi}\right)}d\left({}_aD^{\alpha}_{x_j}\bar{\psi}\right)\end{cases} \quad (25)$$

By comparing equation (24) with equation (25) we get the following Hamilton's equation of motion

$$\begin{cases}\frac{\partial \mathcal{H}}{\partial \psi} = -{}_aD^{\alpha}_t\left(\frac{\partial \mathcal{L}}{\partial\left({}_aD^{\alpha}_t\psi\right)}\right) - {}_aD^{\alpha}_{x_j}\left(\frac{\partial \mathcal{L}}{\partial\left({}_aD^{\alpha}_{x_j}\psi\right)}\right) & (26) \\ \frac{\partial \mathcal{H}}{\partial \pi_{\psi}} = {}_aD^{\alpha}_t\psi & (27) \\ \frac{\partial \mathcal{H}}{\partial\left({}_aD^{\alpha}_{x_j}\psi\right)} = -\frac{\partial \mathcal{L}}{\partial\left({}_aD^{\alpha}_{x_j}\psi\right)} & (28)\end{cases}$$

$$\begin{cases}\frac{\partial \mathcal{H}}{\partial \phi} = -{}_aD^{\alpha}_t\left(\frac{\partial \mathcal{L}}{\partial\left({}_aD^{\alpha}_t\phi\right)}\right) - {}_aD^{\alpha}_{x_j}\left(\frac{\partial \mathcal{L}}{\partial\left({}_aD^{\alpha}_{x_j}\phi\right)}\right) & (29) \\ \frac{\partial \mathcal{H}}{\partial \pi_{\phi}} = {}_aD^{\alpha}_t\phi & (30) \\ \frac{\partial \mathcal{H}}{\partial\left({}_aD^{\alpha}_{x_j}\phi\right)} = -\frac{\partial \mathcal{L}}{\partial\left({}_aD^{\alpha}_{x_j}\phi\right)} & (31)\end{cases}$$

$$\begin{cases}\frac{\partial \mathcal{H}}{\partial \phi^{\dagger}} = -{}_aD^{\alpha}_t\left(\frac{\partial \mathcal{L}}{\partial\left({}_aD^{\alpha}_t\phi^{\dagger}\right)}\right) - {}_aD^{\alpha}_{x_j}\left(\frac{\partial \mathcal{L}}{\partial\left({}_aD^{\alpha}_{x_j}\phi^{+}\right)}\right) & (32) \\ \frac{\partial \mathcal{H}}{\partial \pi_{\phi^{\dagger}}} = {}_aD^{\alpha}_t\phi^{\dagger} & (33) \\ \frac{\partial \mathcal{H}}{\partial\left({}_aD^{\alpha}_{x_j}\phi^{\dagger}\right)} = -\frac{\partial \mathcal{L}}{\partial\left({}_aD^{\alpha}_{x_j}\phi^{+}\right)} & (34)\end{cases}$$

$$\begin{cases}\frac{\partial \mathcal{H}}{\partial \bar{\psi}} = -{}_aD^{\alpha}_t\left(\frac{\partial \mathcal{L}}{\partial\left({}_aD^{\alpha}_t\bar{\psi}\right)}\right) - {}_aD^{\alpha}_{x_j}\left(\frac{\partial \mathcal{L}}{\partial\left({}_aD^{\alpha}_{x_j}\bar{\psi}\right)}\right) & (35) \\ \frac{\partial \mathcal{H}}{\partial \pi_{\bar{\psi}}} = {}_aD^{\alpha}_t\bar{\psi} & (36) \\ \frac{\partial \mathcal{H}}{\partial\left({}_aD^{\alpha}_{x_j}\bar{\psi}\right)} = -\frac{\partial \mathcal{L}}{\partial\left({}_aD^{\alpha}_{x_j}\bar{\psi}\right)} & (37)\end{cases}$$

The conjugate momenta define as:

$$\pi_{\psi} = \frac{\partial \mathcal{L}}{\partial\left({}_aD^{\alpha}_t\psi\right)}, \quad \pi_{\phi} = \frac{\partial \mathcal{L}}{\partial\left({}_aD^{\alpha}_t\phi\right)}, \quad \pi^{\dagger} = \frac{\partial \mathcal{L}}{\partial\left({}_aD^{\alpha}_t\phi^{\dagger}\right)}, \quad \pi_{\bar{\psi}} = \frac{\partial \mathcal{L}}{\partial\left({}_aD^{\alpha}_t\bar{\psi}\right)} \quad (38)$$

After some mathematical manipulation we get the fractional Hamiltonian density

$$\mathcal{H} = \pi_{\psi}\, {}_aD_t^{\alpha}\psi + \pi_{\phi}\, {}_aD_t^{\alpha}\phi + \pi_{\phi^+}\, {}_aD_t^{\alpha}\phi^+ + \pi_{\bar{\psi}}\, {}_aD_t^{\alpha}\bar{\psi} - \frac{1}{2}\bar{\psi} i\gamma^0\, {}_aD_t^{\mu}\psi - \frac{1}{2}\bar{\psi} i\gamma^i\, {}_aD_{x_j}^{\mu}\psi - \frac{1}{2} i\psi\gamma^0\, {}_aD_t^{\mu}\bar{\psi} + \frac{1}{2}\psi i\gamma^i\, {}_aD_{x_j}^{\mu}\bar{\psi} + g_{\pi}\psi\phi\bar{\psi} - \frac{1}{2}\, {}_aD_t^{\mu}\phi\, {}_aD_t^{\mu}\phi^+ + \frac{1}{2}\, {}_aD_{x_j}^{\mu}\phi\, {}_aD_{x_j}^{\mu}\phi^+ + \frac{1}{2}m^2\phi\phi^+ + \frac{1}{4}\lambda^2\phi^2\phi^{+2} \quad (39)$$

So the equations of motion becomes

$$\left(g_{\pi}\phi + i\gamma^0\, {}_aD_t^{\alpha} + i\gamma^i\, {}_aD_{x_i}^{\alpha}\right)\bar{\psi} = 0 \quad (40)$$

$$\left({}_aD_t^{\alpha}\, {}_aD_t^{\alpha} + {}_aD_{x_i}^{\alpha}\, {}_aD_{x_i}^{\alpha} + m^2 + \lambda^2\phi\ \ \phi^+\right)\phi^+ + 2g_{\pi}\psi\bar{\psi} = 0 \quad (41)$$

$$\left({}_aD_t^{\alpha\beta}\, {}_aD_t^{\alpha} + {}_aD_{x_i}^{\alpha}\, {}_aD_{x_i}^{\alpha} + m^2 + \lambda^2\phi\ \ \phi^+\right)\phi = 0 \quad (42)$$

$$\left(i\gamma^0\, {}_aD_t^{\alpha}\psi + i\gamma^i\, {}_aD_{x_i}^{\alpha}\psi - g_{\pi}\psi\phi\right) = 0 \quad (43)$$

$$\left(\square + m^2 + \lambda^2\phi\ \ \phi^+\right)\phi^+ + 2g_{\pi}\psi\bar{\psi} = 0$$

$$\left(\square + m^2 + \lambda^2\phi\ \ \phi^+\right)\phi = 0$$

$$\left(g_{\pi}\phi + i\gamma^{\nu}\partial_{\nu}\right)\bar{\psi} = 0$$

$$\left(i\gamma^{\mu}\partial_{\mu} - g_{\pi}\phi\right)\psi = 0$$

As $\alpha = 1$ we obtain

CONCLUSION

Linear Sigma Model Lagrangian can be written in fractional form in four- dimensional using Riemann-Liouville fractional derivatives.

For a given Lagrangian density we observed that both fractional Euler Lagrange and fractional Hamiltonian equations generate the same results in the two procedures that are used in this paper. The classical results are generated as a specific case of the fractional formulation.

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