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γ^s-CONNECTED SPACES

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ABSTRACT

We define and discuss γ^s-connected spaces, γ^s-components in a space X and γ^s-locally connected spaces.

Keywords: γ-closed(open), γ-closure, γ-regular (open), (γ,β)-continuous (closed, open) function, γ^s-connected space, γ^s-components and γ^s-locally connected spaces.

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INTRODUCTION

Kasahara, 1979 defined an operation α on topological spaces and studied α-closed graphs of a function. Jankovic (1983) defined α- closed set and further worked on functions with α- closed graphs. Ogata (1991) introduced separation axioms γ-T_i ,i = 0,1/2,1,2; and studied some of their properties. Rehman and Ahmad (1992) (respt. Ahmad and Rehman, 1993) defined and discussed several properties of γ-interior, γ-closure, γ-exterior and γ-boundary and (γ,β)- closed (open) mappings in topological spaces (respt. in product spaces), and further investigated the characterizations of (γ,β)-continuous and (γ,β)- closed (open) mappings . Ahmad and Hussain, 2003 (respt. Ahmad and Hussain, 2005) continued studying the properties of γ-operations (respt. γ_0-compact,γ^s- regular and γ^s-normal spaces) in topological spaces. In this paper, we define γ^s- connected spaces and study their properties in topological spaces.

Hereafter we shall write space in place of topological space. We recall some definitions and results used in this paper to make it self-contained.

Definition (Ogata, 1991). Let (X,τ) be a space. An operation $\gamma : \tau \rightarrow P(X)$ is a function from τ to the power set of X such that $V \subseteq V^\gamma$, for each $V \in \tau$, where V^γ denotes the value of γ at V. The operations defined by γ(G) = G, γ(G) = cl(G) and γ(G) = intcl(G) are examples of operation γ.

Definition (Rehman and Ahmad, 1992). Let $A \subseteq X$. A point a∈A is said to be γ-interior point of A iff there exists an open nbd N of a such that $N^\gamma \subseteq A$ and we denote the set of all such points by int_γ (A). Thus $int_\gamma (A) = \{ x \in A : x \in N \in \tau \text{ and } N^\gamma \subseteq A \} \subseteq A$.

Note that A is γ-open (Ogata, 1991) iff A =int_γ (A). A set A is called γ- closed (Ogata, 1991) iff X–A is γ-open.

Definition (Rehman and Ahmad, 1992). A point x∈X is called a γ-closure point of $A \subseteq X$, if $U^\gamma \cap A \neq \emptyset$, for each open nbd. U of x. The set of all γ-closure points of A, is called γ-closure of A and is denoted by cl_γ(A). A subset A of X is called γ-closed, if $cl_\gamma(A) \subseteq A$.

Note that cl_γ(A) is contained in every γ-closed superset of A.

Definition (Ogata, 1991). An operation γ on τ is said to be regular, if for any open nbds U , V of x ∈X, there exists an open nbd W of x such that $U^\gamma \cap V^\gamma \supseteq W^\gamma$.

Definition (Ogata, 1991). An operation γ on τ is said to be open if for every nbd U of each x∈X, there exists γ-open set B such that x∈B and $U^\gamma \supseteq B$.

1. γ^s-connected space

Definition 1. A topological space X is said to be γ^s-connected if there does not exist a pair A , B of nonempty disjoint open subset of X such that $X = A^\gamma \cup B^\gamma$, otherwise X is called γ^s- disconnected. In this case, the pair (A, B) is called a γ^s- disconnection of X. A subset A of a space X is γ^s- connected if it is γ^s-connected as a subspace.

Example 1. Let X = {a,b,c}, τ = { ∅, X , {a},{b},{a,b}}. For b∈ X, define an operation $\gamma : \tau \rightarrow P(X)$ such that

$$\gamma(A) = \begin{cases} A \quad , & \text{if } b \in A \\ \text{clintcl(A)}, & \text{if } b \in A \end{cases}$$

Then X is connected. But X is not γ^s- connected, because there exists a pair {a}, {a,b} of open sets such that $\{a\}^\gamma \cup \{b\}^\gamma = X$, and $\{a\}^\gamma \cap \{b\}^\gamma = \emptyset$.

Example 2. Let X = {a,b,c}, τ = { ∅, X , {a},{b},{a,b}}. Define an operation $\gamma : \tau \rightarrow P(X)$ such that γ (A) = intcl(A).

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Then X is γ^s- connected, since there does not exists a pair A, B of open sets such that $X = A^{\gamma} \cup B^{\gamma}$ and $A^{\gamma} \cap B^{\gamma} = \varnothing$.

Example 3. Any infinite space with co-finite γ-topology is γ^s- connected, if γ is γ-open.

Example 4. Every γ-indiscrete space is γ^s- connected.

We give the characterization of γ^s-connected space, the proof of which is straight forward.

Definition (Ahmad and Hussain, 2005). An operation $\gamma: \tau \to P(X)$ is said to be strictly regular, if for any open nbds U , V of $x \in X$, there exists an open nbd W of x such that $U^{\gamma} \cap V^{\gamma} = W^{\gamma}$.

Definition (Ahmad and Hussain, 2005). An operation $\gamma: \tau \to P(X)$ is said to be γ-open, if V^{γ} is γ-open for each $V \in \tau$.

Example 1. Let X={a,b,c}, τ ={ ∅, X, {a}, {b},{a,b}}. Define an operation $\gamma : \tau \to P(X)$ by γ(A) = intcl(A).

Clearly the γ-open sets are only ∅, X, {a}, {b},{a,b}. It is easy to see that γ is strictly regular and γ-open on X.

Example 2. Let X={a,b,c}, τ ={ ∅, X, {a}, {b},{a,b}}. Define an operation $\gamma : \tau \to P(X)$ by γ(A) = cl(A).

Clearly the γ-open sets are only ∅, X. It is easy to see that γ is strictly regular but not γ-open on X.

Theorem 1. A topological space X is γ^s -disconnected (respt. γ^s **-**connected) iff there exists (respect. does not exist) nonempty subset A of X such that A^{γ} is both γ-open and γ-closed in X .

Definition (Ogata, 1991). A mapping $f : (X, \tau_1) \to (Y, \tau_2)$ is said to be (γ,β)-continuous if for each $x \in X$ and each open set V containing f (x), there exists an open set U such that $x \in U$ and $f(U^{\gamma}) \subseteq V^{\beta}$, where $\gamma : \tau_1 \to P(X)$; $\beta : \tau_2 \to P(Y)$ are operations on τ_1 and τ_2 respectively.

A (γ,β)-continuous mapping (Rehman and Ahmad, 1992) has been characterized as :

If $f : (X, \tau_1) \to (Y, \tau_2)$ is a mapping and β is open, then f is (γ,β)-continuous if f for each open set V in Y, $f^{-1}(V)$ is γ-open in X. We use this characterization and prove:

Theorem 2. The (γ, β)-continuous image of γ^s-connected space is γ^s-connected, where β is open and γ-open and γ is strictly regular and γ-open.

Poof. Let f: $(X, \tau_1) \to (Y, \tau_2)$ be (γ, β)-continuous from a γ^s-connected space (X, τ_1) onto a space (Y, τ_2). Suppose that Y is γ^s-disconnected and (A, B) is a γ^s-disconnection of Y. Since f is (γ,β)-continuous, where β be open operation and A^{β} and B^{β} are both β-open in Y. Therefore $f^{-1}(A^{\beta})$, $f^{-1}(B^{\beta})$are both γ-open in X. Also

$$(f^{-1}(A^{\beta}))^{\gamma} \cup (f^{-1}(B^{\beta}))^{\gamma} = (f^{-1}(A^{\beta} \cup B^{\beta}))^{\gamma} = (f^{-1}(Y))^{\gamma} = (X)^{\gamma} = X,$$

and $$(f^{-1}(A^{\beta}))^{\gamma} \cap (f^{-1}(B^{\beta}))^{\gamma} = (f^{-1}(A^{\beta} \cap B^{\beta}))^{\gamma} = (f^{-1}(\varnothing))^{\gamma} = (\varnothing)^{\gamma} = \varnothing.$$

Hence ($f^{-1}(A^{\beta})$, $f^{-1}(B^{\beta})$) is a pair of γ^s-disconnection of X. This contradiction shows that Y is γ^s- connected. This completes the proof.

Next, we characterize γ^s -connectedness in terms of γ-boundary as:

Theorem 3. A space X is γ^s-connected iff every nonempty proper subspace A such that A^{γ} has a nonempty γ-boundary, where γ is γ-open.

Proof. Suppose that a nonempty proper subspace A of a γ^s-connected space X such that A^{γ} has empty γ-boundary. Then A is γ-open and $cl_{\gamma}(A^{\gamma}) \cap cl_{\gamma}(X-A^{\gamma}) = \varnothing$. Let p be a γ-limit point of A^{γ}. Then $p \in cl_{\gamma}(A^{\gamma})$ but $p \notin cl_{\gamma}(X-A^{\gamma})$. In particular $p \notin X-A^{\gamma}$ and so $p \in A^{\gamma}$. Thus A^{γ} is γ-closed and γ-open. By theorem 1, X is γ^s-disconnected. This contradiction proves that A has a nonempty γ-boundary.

Conversely, suppose X is γ^s-disconnected. Then by theorem 1, X has a proper subspace A such that A^{γ} is both γ-closed and γ-open. Then $cl_{\gamma}(A^{\gamma}) = A^{\gamma}$, $cl_{\gamma}(X-A^{\gamma}) = (X-A^{\gamma})$ and $cl_{\gamma}(A^{\gamma}) \cap cl_{\gamma}(X-A^{\gamma}) = \varnothing$. So A has empty γ-boundary, a contradiction. Hence X is γ^s- connected. This completes the proof.

Definition (Ahmad and Hussain, 2005**)**. Let X be a space and $A \subseteq X$. Then the class of γ-open sets in A is defined in a natural way as : $\tau\gamma_A = \{ A \cap O : O \in \tau_{\gamma} \}$, where τ_{γ} is the class of γ-open sets of X. That is, G is γ-open in A iff $G = A \cap O$, where O is a γ-open set in X.

Theorem 4. Let (A,B) be a γ^s-disconnection of a space X and C^{γ} be a γ^s-connected subspace of X. Then C is contained in A or B, where γ is strictly regular.

Proof. Suppose that C is neither contained in A nor in B. Then $A \cap C$, $B \cap C$ are both nonempty open subsets of C such that

$$\begin{aligned}(A \cap C)^{\gamma} \cap (B \cap C)^{\gamma} &= (A^{\gamma} \cap C^{\gamma}) \cup (B^{\gamma} \cap C^{\gamma}) \quad (\gamma \text{ is strictly regular}) \\ &= (A^{\gamma} \cap B^{\gamma}) \cap C^{\gamma} \\ &= \varnothing \cap C^{\gamma} = \varnothing\end{aligned}$$

and $(A \cap C)^{\gamma} \cup (B \cap C)^{\gamma} = (A^{\gamma} \cap C^{\gamma}) \cup (B^{\gamma} \cap C^{\gamma})$

$= (A^{\gamma} \cup B^{\gamma}) \cap C^{\gamma}$
$= X \cap C^{\gamma} = C^{\gamma}$.

This gives that $(C \cap A, C \cap B)$ is a γ^s-disconnection of C^{γ}. This contradiction proves the theorem.

Theorem 5. Let $X = \bigcup_{\alpha \in I} X_{\alpha}$,where each X^{γ}_{α} is γ^s-connected and $\bigcap_{\alpha \in I} X_{\alpha} \neq \emptyset$. Then X is γ^s-connected, where γ is strictly regular.

Proof. Suppose on the contrary that (A, B) is a γ^s-disconnection of X. Since each X^{γ}_{α} is γ^s-connected, therefore by theorem 4, $X_{\alpha} \subseteq A$ or $X_{\alpha} \subseteq B$. Since $\cap X_{\alpha} \neq \emptyset$, therefore all X_{α} are contained in A or in B. This gives that, if $X \subseteq A$, then $B = \emptyset$ or if $X \subseteq B$, then $A = \emptyset$. This contradictions proves that X is γ^s-connected. Hence the proof.

Using theorem 5, we characterize γ^s-connectedness as:

Theorem 6. A space X is γ^s-connected iff for every pair of points x,y in X , there is a γ^s-connected subset of X which contains both x and y, where γ is strictly regular.

Proof. The necessity is immediate since the γ^s-connected space itself contains these two points.

For the sufficiency, suppose that for any two points x,y ; there is a γ^s-connected subspace $C^{\gamma}_{x,y}$ of X such that x, $y \in C^{\gamma}_{x,y}$. let $a \in X$ be a fixed point and $\{C^{\gamma}_{a,x} , x \in X\}$ be a class of all γ^s-connected subsets of X which contain a and $x \in X$. Then $X = \bigcup_{x \in X} C_{a,x}$ and $\bigcap_{x \in X} C_{a,x} \neq \emptyset$. Therefore by theorem 5, X is γ^s-connected. This completes the proof.

Theorem 7. Let C be a γ^s-connected subset of a space X and $A \subseteq X$ such that $C \subseteq A \subseteq Cl_{\gamma}(C^{\gamma})$. Then A is γ^s-connected, where γ is strictly regular.

Proof. It is sufficient to show that $cl_{\gamma}(C^{\gamma})$ is γ^s- connected. On the contrary, suppose that $cl_{\gamma}(C)$ is γ^s- disconnected. Then there exists a γ- disconnection (H, K) of $cl_{\gamma}(C)$. That is, there are $H \cap C$, $K \cap C$ open ses in C such that

$$
\begin{aligned}
(H \cap C)^{\gamma} \cap (K \cap C)^{\gamma} &= (H^{\gamma} \cap C^{\gamma}) \cap (K^{\gamma} \cap C^{\gamma}) \\
&= (H^{\gamma} \cap K^{\gamma}) \cap C^{\gamma} \\
&= \emptyset \cap C^{\gamma} = \emptyset.
\end{aligned}
$$

and

$$
\begin{aligned}
(H \cap C)^{\gamma} \cup (K \cap C)^{\gamma} &= (H^{\gamma} \cap C^{\gamma}) \cup (K^{\gamma} \cap C^{\gamma}) \\
&= (H^{\gamma} \cup K^{\gamma}) \cap C^{\gamma} \\
&= cl_{\gamma}(C^{\gamma}) \cap C^{\gamma} = C^{\gamma}.
\end{aligned}
$$

This gives that $(H \cap C, K \cap C)$ is a γ^s- disconnection of C^{γ}, a contradiction. This proves that $cl_{\gamma}(C)$ is γ^s-connected.

2. γ^s-Components in space X

Definition 4. A maximal γ- connected subset of a space X is called a γ- component of X. If X is itself γ^s-connected, then X is the only γ^s- component of X.

Next we study the properties of γ-components of a space X :

Theorem 8. Let X be a space. Then

(1) for each $x \in X$,there is exactly one γ^s-component of X containing x, where γ is strictly regular.
(2) each γ^s-connected subset of X is contained in exactly one γ^s-component of X.
(3) a γ^s-connected subset of X which is both open and closed is γ^s-component , if γ is strictly regular.
(4) each γ^s-component of X is γ-closed in X, where γ is strictly regular.

Proof. (1) Let $x \in X$ and $\{C^{\gamma}_{\alpha} : \alpha \in I\}$ a class of all γ^s-connected subsets of X containing x. Put $C = \bigcup_{\alpha \in I} C_{\alpha}$, then by theorem 5, C is γ^s -connected and $x \in X$. Suppose $C \subseteq C^*$ for some γ^s-connected subset C^* of X. Then $x \in C^*$ and hence C^* is one of the C_{α}'s and hence $C^* \subseteq C$. Consequently $C = C^*$. This proves that C is a γ-component of X which contains x.

(2) Let A be a γ^s-connected subset of X which is not a γ^s-component of X. Suppose that C_1, C_2 are γ^s-components of X such that $A \subseteq C_1$, $A \subseteq C_2$. Since $C_1 \cap C_2 = \emptyset$, $C_1 \cup C_2$ is another γ^s-connected set which contains C_1 as well as C_2, a contradiction to the fact that C_1 and C_2 are γ^s-components. This proves that A is contained in exactly one γ^s- component of X

(3) Suppose that A is γ^s-connected subset of X such that A^{γ} is both γ-open and γ-closed. By (2), A is contained is exactly one γ^s-component C of X. If A is a proper subset of C, and since γ is strictly regular, therefore

$$(C \cap A)^{\gamma} \cup (C \cap (X-A))^{\gamma} = C^{\gamma},$$

and

$$(C \cap A)^{\gamma} \cap (C \cap (X-A))^{\gamma} = \emptyset.$$

Thus $(C \cap A, C \cap (X-A))$ is a γ^s-disconnection of C^{γ}, a contradiction. Thus A = C.

(4) Suppose a γ^s-component C of X is not γ-closed. Then by Theorem 7, $cl_{\gamma}(C)$ is γ^s-connected containing γ^s-component C of X. This implies $C = cl_{\gamma}(C)$ and hence C is γ-closed. This completes the proof.

3. γ^s-locally connected spaces

Definition 5. A space X is said to be γ^s-locally connected if for any point $x \in X$ and any open set U containing x, there is a connected open set V such that $x \in V$ and $V^{\gamma} \subseteq U$.

Example: Let $X = \{a,b,c\}$, $\tau = \{ \varnothing, X, \{a\},\{b\},\{a,b\}\}$. Define an operation $\gamma : \tau \rightarrow P(X)$ by $\gamma(A) = \text{intcl}(A)$.

Routine calculations give that $\varnothing$, X, {a}, {b}, {a,b} are the only γ-open sets. Clearly X is γ^s-locally connected.

The following theorem shows that γ-locally connectedness is a γ-open hereditary property :

Theorem 9. A γ-open subset of γ^s-locally connected space is γ^s-locally connected.

Proof. Let U be a open subset of a γ^s-locally connected space X. Let $x \in U$ and V a open nbd of x in U. Then V is an open nbd of x in X. Since X is γ^s-locally connected, therefore there exists a connected, open nbd W of x such that $x \in W$ and $W^{\gamma} \subseteq V$. In this way W is also a connected open nbd x in U such that $x \in W$ and $W^{\gamma} \subseteq V \subseteq U$ or $x \in W$ and $W^{\gamma} \subseteq V$.This proves that U is γ^s-locally connected. Hence the proof.

Definition (Ahmad and Rehman, 1993). A mapping $f : (X, \tau_1) \rightarrow (Y, \tau_2)$ is said to be (γ,β)-closed (respt. (γ,β)-open), if for any γ-closed (γ-open) set A of X, f(A) is β-closed (respt. β-open) in Y.

Theorem 10. A (γ,β)-continuous (γ,β)-open surjective image of γ^s-locally connected space is a γ^s-locally connected space, where β is open and γ-open and γ is γ-open.

Proof. Let $f: (X, \tau_1) \rightarrow (Y, \tau_2)$ be (γ,β)-continuous, (γ,β)-open from a γ^s-locally connected space X to a space Y. We show that Y = f (X) is γ^s-locally connected space. Let $y \in Y$ and choose $x \in X$ such that f (x) = y. Let U^{β} be β-open set containing y, then $f^{-1}(U^{\beta})$ is γ-open in X and $x \in f^{-1}(U^{\beta})$. Since X is γ^s-locally connected, therefore there exists a connected, open set V containing x such that

$$x \in V^{\gamma} \subseteq f^{-1}(U^{\beta}).$$

This gives that $f(x) \in f(V^{\gamma}) \subseteq f f^{-1}(U^{\beta}) = U^{\beta}$ or $y \in f(V^{\gamma}) \subseteq U^{\beta}$. Since f is (γ,β)-continuous, therefore $f(V^{\gamma})$ is γ-open. Moreover $f(V^{\gamma})$ is γ^s-connected. This proves that Y is γ^s-locally connected. Hence the proof.

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